A study on stability results for stochastic nonlinear systems of difference equations

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Abstract
In this paper, we present some results for the asymptotic stability of solutions for nonlinear fractional difference equations involving Riemann-Liouville-like difference operator. The results are obtained by using Krasnoselskii’s fixed point theorem and discrete Arzela-Ascoli’s theorem. Three examples are also provided to illustrate our main results.

Keywords: Nonlinear fractional difference, asymptotic stability solution, Riemann-Liouville-like difference operator.

Introduction
We consider the asymptotic stability of solutions for nonlinear fractional difference equations:

\[ \Delta^\alpha x(t) = f(t + \alpha, x(t + \alpha)), t \in N_0, 0 < \alpha \leq 1, \]

\[ \Delta^{-1} x(t)|_{t=0} = x_0. \]

where \( \Delta^\alpha \) is a Riemann-Liouville-like discrete fractional difference, \( f((0_\nu + \alpha) x \mathbb{R} \rightarrow \mathbb{R} \) is continuous with respect to \( t \) and \( x \), \( N_\alpha = \{0, a + 1, a + 2, \ldots\} \).

Fractional differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [1]. Most of the present works were focused on fractional differential equations, and the references there in. However, very little progress has been made to develop the theory of the analogous fractional finite difference equation [3].

Due to the lack of geometry interpretation of the fractional derivatives, it is difficult to find a valid tool to analyze the stability of fractional difference equations [5]. In the case that it is difficult to employ Liapunov’s direct method, fixed point theorems are usually considered in stability. Motivated by this idea, in this paper, we discuss asymptotic stability of nonlinear fractional difference equations by using Krasnoselskii’s fixed point theorem and discrete Arzela-Ascoli’s theorem. Examples are provided to illustrate the main results.

We introduce preliminary facts of discrete fractional calculus.

Preliminaries

Definition 2.1
Let \( v > 0 \). The \( v \)-th fractional sum \( x \) is defined by

\[ \Delta^v f(x) = \frac{1}{\Gamma(v)} \sum_{n=0}^{t} (t - n - 1)^{(v-1)} f(x'). \]

(2.1)

where \( f \) is defined for \( x = a \mod (1) \) and \( \Delta^v f \) is defined for \( t = (a + 1) \mod (1) \), and \( t^{(v)} = \frac{t^v}{\Gamma(v+1)} \). The fractional sum \( \Delta^v \) maps functions defined on \( N_0 \) to functions defined on \( N_a \).
Definition 2.2 Let \( \mu = 0 \) and \( m = 1 < \mu < m \), where \( m \) denotes a positive integer, \( \lceil \mu \rceil \) ceiling of number. Set \( \psi = m - \mu \). The \( \mu \)-th fractional difference is defined as
\[
\Delta^\mu f(x) = \Delta^{m-\mu} f(x) = \Delta^m (\Delta^{-\mu} f(x))
\]
(2.2)

Let \( f \) be real – value function defined on \( \mathbb{N} \), then the following equalities hold:
\[
\left(\Delta^\mu, \Delta^{-\mu} \right) f(x) = \Delta^{-\mu} \left(\Delta^\mu \right) f(x) = \Delta^{-\mu} \left(\Delta^{-\mu} \right) f(x) = \frac{(x - a)^{\mu - 1}}{\Gamma(\mu)} f'(a).
\]

Stochastic Non-Linear System

We say that \( S_{\mu} \) is stable in probability if for any \( a = 0 \) and any \( \epsilon < 1 \), there exist a \( \delta \) such that for all \( t \in [0, T] \) and \( |x(0)| < \delta \), the following conditions holds:
\[
P\left( |x(t)| > \epsilon \right) < 1 - \delta
\]
(3.4)

In the following we consider the Stochastic Non-linear System (SNS)
\[
dx = \left( f(x, t) dt + h(x, t) u(t) + g(x, t) dw_t \right)
\]
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( f(x, t) \), \( h(x, t) \) \( g(x, t) \) are continuous and Lipschitz functions with \( f(t, 0) = 0 \) and \( g(t, 0) = 0 \).

This extension of 3.2.1 provided in that guarantees the existence of a \( C^{\infty} \) control law \( u = \theta(x, t) \) in such a way that \( S_{\mu} \), satisfy the stability in
\[
dx = \left( f(x, t) dt + h(x, t) \theta(x, x) + g(x, t) dw_t \right)
\]

Denote by \( D \) the infinitesimal generator of the stochastic process solution of the uncontrolled part of SNS equation 3.2, that is, \( D \) is the second order differential operator defined for any function \( \Phi \in C^{2,1}(\mathbb{R}^n, \mathbb{R}) \) by
\[
D\Phi(x, t) = \frac{\partial \Phi(x, t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial^2 \Phi(x, t)}{\partial x_i^2} + \sum_{i=1}^{n} \frac{\partial \Phi(x, t)}{\partial x_i} h_i(x, t) + \sum_{i=1}^{n} \frac{\partial \Phi(x, t)}{\partial x_i} g_i(x, t)
\]

For any \( \theta \in C^{2,1}(\mathbb{R}^n, \mathbb{R}) \), denote by \( D_x \) the first order differential operator for any function \( \Phi \in C^{2,1}(\mathbb{R}^n, \mathbb{R}) \) by
\[
D_x \Phi(x, t) = \sum_{i=1}^{n} \frac{\partial \Phi(x, t)}{\partial x_i} h_i(x, t) + \sum_{i=1}^{n} \frac{\partial \Phi(x, t)}{\partial x_i} g_i(x, t)
\]

Define \( X \) as the infinitesimal generator for the stochastic process solution of the closed-loop system 3.3, that is, \( X \) is the differential operator defined for any function \( \Phi \) in \( C^{2,1}(\mathbb{R}^n, \mathbb{R}) \) by
\[
X \Phi(x, t) = D_x \Phi(x, t) + \sum_{i=1}^{n} D_x \Phi(x, t) u_i
\]

In the following we extend the concept of stochastic Lyapunov condition introduced in definition provided used for stability in probability of SNS equation 3.2 at the neighborhood of the origin.

Application

In this section we illustrate our results by a designing a numerical example.

First Example

Consider the Stochastic Non-Linear System
\[
d\left( \frac{x_1}{x_2} \right) = \left( x_1 + \frac{x_2^2}{2} \right) dt + \left( \frac{x_1}{2} \right) dw_t
\]
(4.1)

Where \( \{w_t\} \) a standard real-valued Wiener process, \( x \) is a real-valued measurable control law,
\[
f_1(x, u, x_2) = x_1 + x_2^2, f_2(x, u, x_2) = 2,
\]
\[
h_1(x, u, x_2) = 0, h_2(x, u, x_2) = 1,
\]
\[
g_1(x, u, x_2) = x_2 + \frac{1}{2}
\]
Define the Lyapunov function in the form
\[
\Phi(x, t, x_2) = 2x_2^2 \exp(2t) + (x_2 + x_2 \exp(t))^2
\]

A simple calculation shows that
\[
\frac{d\Phi(x, t, x_2)}{dx_2} = 0 \Rightarrow \frac{d\Phi(x, t, x_2)}{dx_2} = -x_2 \exp(t)
\]
(4.2)

Therefore,
\[
\Phi(x, t, x_2) = -\frac{x_2^2}{2} \exp(-t)
\]
\[
\Phi(x, t, x_2) = -\frac{x_2^2}{2} \exp(-t)
\]

The later inequality implies that are fulfilled with \( \theta(x) = \frac{1}{2} x^2 \) and \( \phi(x) = \frac{1}{2} x^2 \), and by theorem there exist a \( C^{\infty} \) feedback law \( \Phi(x, t, x_2) \) with \( \Phi(x, 0, 0) \) such that \( S_{\mu} \) is stable in probability with respect to the resulting closed-loop system deduced from 4.1.

Homogeneous Difference Equations

When the number of lags grows large (3 or greater), solving linear difference equations by substitution is tedious. The key to understanding linear difference equations is the study of the homogeneous portion of the equation [8]. In the general linear difference equation,
\[
X_1 = \delta_0 + \delta_1 x_{t+1} + \delta_2 x_{t+2} + \cdots + \delta_n x_{t+n}
\]

The homogenous portion is defined as the terms involving only \( y \),
\[
X_1 = \delta_0 x_{t+1} + \delta_2 x_{t+2} + \cdots + \delta_n x_{t+n}
\]
(4.4)

The intuition behind studying this portion of the system is that, given the sequence of \( \{x(t)\} \), all of the dynamics and the
stability of the system are determined by the determined by
the relationship between contemporaneous and it’s lagged
values which allows the determination of the parameter
values where the system is stable \[15\]. Again, consider the
homogeneous portions of the simple 1st order system

\[ y_t = \alpha_1 y_{t-1} + \epsilon_t \]

Which has homogenous portion

\[ y_t = \alpha_1 y_{t-1} \]

It is easy to show that

\[ y_t = \alpha_1 y_{t-1} \]

Is also a solution by examining the solution to the linear
difference equation. The solution of the form for an
arbitrary constant c.

\[ y_t = c \alpha_1^t \]

\[ y_{t-1} = c \alpha_1^{t-1} \]

and

\[ y_t = \alpha_1 y_{t-1} \]

Putting these two together shows that

\[ y_t = \alpha_1 y_{t-1} \]

This is also a solution by examining the solution to the linear
difference equation. The solution of the form \( c \alpha_1^t \)
for an arbitrary constant c.

\[ y_t = c \alpha_1^t \]

\[ y_{t-1} = c \alpha_1^{t-1} \]

Linear Homogeneous Difference Equations
An equation of the form

\[ y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = 0 \]

in which \( \alpha_1 \) and \( \alpha_2 \) are constants, is a linear second-order
difference equation, with constant coefficients. This is
precisely the type of equation we found for \( y_t \) in the
previous section. When \( k=0 \), we have the homogeneous
equation

\[ y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = 0 \]

It follows that if we know two solutions \( y_t^{(1)} \) and \( y_t^{(2)} \)
of the difference equation, then

\[ A y_t^{(1)} + B y_t^{(2)} \]

is also a solution for any constants A and B. Suppose we
are given a homogeneous difference equation

\[ y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = 0 \]

In order to determine the sequence of values \( y_t \) we must
know the initial values \( y_0 \) and \( y_1 \). Given these values, \( y_2 \)
is determined by the equation with \( t=2 \), \( y_3 \) is then determined
by the equation with \( t=3 \) and so on. So if we are looking for a
solution \( y_t = A y_t^{(1)} + B y_t^{(2)} \), we have to choose A and B
so that the formula fits the initial conditions when \( t=0 \) and
\( t=1 \). These two conditions determine appropriate values
for the two arbitrary constants \[15\]. This means that the general
solution of the homogeneous difference equation is given by
the formula displayed above.

We shall now describe a practical method for finding two
solutions \( y_t^{(1)} \) and \( y_t^{(2)} \), based on the auxiliary equation

\[ z^2 + \alpha_1 z + \alpha_2 = 0 \]

This is a quadratic equation. We observed that such an
equation may have two distinct solutions, or just one
solution, or no solutions, depending on the value of the quantity \( \alpha_1^2 - 4 \alpha_2 \).

The auxiliary equation has just one solution with another
example
It is clear that we cannot get a solution involving two
arbitrary constants by the method used above. If the (one)
solution of the auxiliary equation is \( \alpha \), then \( y_t = \alpha^t \) is a
solution of the difference equation as before, but we need to
find another.

The auxiliary equation has exactly one solution when

\[ \alpha_1^2 - 4 \alpha_2 = 0 \]

that is, when \( \alpha_2 = \frac{\alpha_1^2}{4} \). Then the equation

\[ z^2 + \alpha_1 z + \alpha_2 = 0 \]

Can be written in the form

\[ z^2 + \alpha_1 z + \frac{\alpha_1^2}{4} = 0 \]

And the (one) solution is \( \alpha = -\frac{\alpha_1}{2} \). We claim that a second
solution of the difference equation is \( y_t = \alpha^t \). Substituting
this,

\[ y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = 0 \]

Because \( \alpha \) satisfies the auxiliary equation, we have

\[ \alpha^2 + \alpha_1 \alpha + \alpha_2 = 0 \]

Furthermore, since \( \alpha = -\frac{\alpha_1}{2} \)
and

\[ \alpha_1^2 - 4 \alpha_2 = 0 \]

it follows that

\[ \alpha_1 + 2 \alpha_2 = -\frac{\alpha_1^2}{2} + 2 \alpha_2 = 0 \]

Hence \( \alpha^2 \) is a
solution, as claimed.

The general solution is therefore

\[ y_t = C \alpha^t + D \alpha^t = (C + D) \alpha^t \]

The values of the constants C and D can be determined by
using the initial values \( y_0 \) and \( y_1 \).

Third Example: Consider the difference equation

\[ 1 \cdot y_t = 1 \cdot y_{t-1} + 6 \cdot y_{t-2} \]

The auxiliary equation is

\[ \alpha^2 - 6 \alpha + 9 = 0 \]

That is (\( \alpha - 3 \))^2 = 0

There is therefore just one solution, \( \alpha = 3 \) of the auxiliary
equation. The general solution to the difference equation is

\( (C+D) \alpha^t \)

Using the facts that \( y_0 = 1 \) and \( y_1 = 1 \)
we must have

\( D = 1, 3(C+D) = 1 \)

So that \( C = -2/3 \) and \( D = 1 \), giving

\[ y_t = \left(-\frac{2}{3} t + 1\right) \alpha^t \]

Conclusion
In this paper we derived Stability in probability of stochastic
nonlinear system there are many types of stochastic system
although they do not trivial solution. We have established the
stability probability of non-trivial solutions for stochastic
nonlinear system. We have derived a stochastic version of
control Lyapunov function and provided the necessary and
sufficient condition in probability of a non-trivial solution for
stochastic non-linear system exists. The numerical examples
are solved to illustrate our results

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