Regular number of line graph of a graph

MH Muddebihal, Abdul Gaffar, Shabbir Ahmed

Abstract

For any (p, q) graph G, a line graph L(G) is obtained from G by taking each edge as a vertex in L(G). The regular number of the L(G) is the minimum number of subsets into which the edge set of L(G) should be partitioned so that the sub graph induced by each subset is regular and is denoted by $r_{r}(G)$. In this paper some results on regular number of $r_{r}(G)$ were obtained and expressed in terms of elements of G.

Keywords: Regular number, Line graph, Domination number, Total domination.

1. Introduction

All graphs considered here are simple, finite, non-trivial. As usual p and q denote the number of vertices and edges of a graph G and, the maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex $v$ is called a cut vertex if removing it from G increases the number of components of G. A graph G is called trivial if it has no edges. The maximum distance between any two vertices in G is called the diameter, denoted by diam (G). A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices is of degree 1 or 3. The path and tree numbers were introduced by Stanton James and Cown in [10]. Any undefined term in this paper may be found in [2]. Let G = (V, E) be a graph. A set $D' \subseteq V$ is said to be a dominating set of G, if every vertex in $(V - D')$ is adjacent to some vertex in $D'$. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. A dominating set is said to be total dominating set of G, if $N(D') = V$ or equivalently, if for every $v \in V$, there exists a vertex $u \in D'$, $u \neq v$, such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. Domination related parameters are now well studied in graph theory. The total domination $\gamma_t(G)$ were studied by M.H. Muddebihal, Srinivasa, G, and A.R. Sedamkar in [6]. A dominating set D of L(G) is a regular total dominating set (RTDS) if the induced sub graph $< D >$ has no isolated vertices and $deg(v) = 1, \forall v \in D$. The regular total domination number $\gamma_{rt}(L(G))$ is the minimum cardinality of a regular total dominating set. The regular total domination in line graphs were studied by M.H. Muddebihal, U.A. Panfarosh and Anil. R. Sedamkar in [7]. A total dominating set in a graph were studied by M.A. Henning and A. Yeo in [9]. Total domination in graphs were studied by E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi in [1]. Total domination and total domination subdivision numbers of graphs were studied by O. Favaron, H. Karami and S.M. Sheikholeslami in [9]. Recent results on total domination in graphs were studied by M. A. Henning in [3]. On matching and total domination in graphs, were studied by M. A. Henning, L. Kang, E. Shan and A. Yeo in [5]. The total (k)-domination number of Cartesian products of graphs were studied by N. Li and X. Hou in [8].

2. Results

The following result is obvious, hence we omit its proof.

**Theorem 1:** For any regular graph G, $r_{r}(G) = 1$.

Next, we obtain the regular number of a path.

**Theorem 2:** For any path $P_p$, $r_{r}(P_p) = 2$. 

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Proof: Let $P_p : e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, \ldots, e_{p-2} = v_{p-2}v_{p-1}, e_{p-1} = v_{p-1}v_p$ be a path. Now, we have, $L(P_p) = \{ e_1, e_2, e_3, e_4, \ldots, e_{p-2}, e_{p-1} \}$ be the edge set of $L(P_p)$. Let $F_1 = \{ e_2, e_3, e_5, e_6, \ldots, e_{p-2}, e_{p-1} \}$ and $F_2 = \{ e_2e_3, e_4e_5, e_6e_7, \ldots, e_{p-2}e_{p-1} \}$ be the minimum regular partition of $L(P_p)$. Hence, $r_l(L(P_p)) = |F_1| = |F_2|$, clearly $r_l(L(P_p)) = 2$.

In the following theorem we establish the regular number of line graph of a binary tree.

**Theorem 3:** For any non-trivial binary tree $T$, $r_l(T) \leq \Delta(T)$.

Where $\Delta(T)$ is the maximum degree of $T$.

**Proof:** Let $T$ be a non-trivial binary tree. Suppose, a binary tree $T$ has $p \geq 5$ vertices, then there exists only $F_1$ and $F_2$ partitions of $L(T)$. Thus, $\Delta(T) \geq r_l(T)$. Suppose a binary tree $T$ has $p \geq 7$ vertices, then the edges incident to the vertex of degree 2 gives a bridge in $L(T)$ and remaining blocks of $L(T)$ are $K_3$. Now, the set of $K_3$'s are partitioned into two sets which are adjacent to each other and another set contains only a bridge of $L(T)$. Hence, in any partition of $L(T)$, we have $F_1$, $F_2$, and $F_3$. Since, $F = \{ F_1, F_2, F_3 \}$ is the minimum regular partition of $r_l(T)$.

Thus, $r_l(T) \leq \frac{1}{3} |F_1, F_2, F_3|$, clearly $r_l(T) \leq 3$ which gives $r_l(T) \leq \Delta(T)$.

In the next result we obtain the regular number of line graph of a complete bipartite graph.

**Theorem 4:** For any complete bipartite graph $K_{m,n}$ , $r_l(K_{m,n}) = 1$.

**Proof:** Let $G = K_{m,n}$ be a complete bipartite graph. Then for any $K_{m,n}$ if $m > n$. Then $L(K_{m,n})$ is $m$-regular which gives only one partition. Hence, $r_l(K_{m,n}) = 1$. For a $K_{m,n}$ if $n > m$. Then $L(K_{m,n})$ is $n$-regular. Thus $r_l(K_{m,n}) = 1$. Further for a $K_{m,n}$ if $m = n$, then $L(K_{m,n})$ is $m$ or $n$-regular. On all partitions of $L(K_{m,n})$, we have $r_l(K_{m,n}) = 1$. Hence, for any $K_{m,n}$, $r_l(K_{m,n}) = 1$.

Next, we obtain the result of regular number of line graph of a complete graph.

**Theorem 5:** For any complete graph $K_p$ , then $r_l(K_p) = 1$.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_p$ be the vertices of $K_p$ and each vertex is of degree $p-1$. Let $e_1, e_2, e_3, \ldots, e_{p(p-1)/2}$ be the edges of $K_p$ such that $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, \ldots, e_{p(p-1)/2} = v_{p-1}v_p$.

Now, in $L(K_p)$ the edge set corresponds to the vertex set. Let $e_1, e_2, e_3, \ldots, e_{p(p-1)/2}$ be the vertices of $L(K_p)$ and every vertex is regular of degree 2(p-2). Hence, the regular partition is one. Thus, $r_l(K_p) = 1$. Further the above proof can be stated in terms of regularity of $G$ such as; since $K_p$ is a regular graph with $p-1$ regular. Also the degree of each edge is $p$-regular. In, $L(G) \ E(G) = V[L(G)]$, hence degree of each vertex in $L(G) = p$. Clearly $L(K_p)$ is $p$-regular. Hence, $r_l(K_p) = 1$.

Now, we obtain the exact value of $r_l(T)$ and the cut vertices.

**Theorem 6:** For any tree with $n \geq 2$ cut vertices with same degree, $r_l(T) = 2$.

**Proof:** Let $T$ be a non-trivial tree with $n$-cut vertices with same degree. In $L(T)$, each block is complete with same regular. Since each cut vertex of $L(T)$ is incident with exactly two blocks. Then each block belongs to different partitions of $L(T)$ which are $F_1$ and $F_2$. Then, $F = \{ F_1, F_2 \}$ such that $|F_1, F_2| = r_l(T) = 2$.

Next, we obtain the result of regular number of line graph of a wheel.

**Theorem 7:** For any wheel $W_p$, with $p \geq 5$ vertices, $r_l(W_p) = 2$.

**Proof:** Let $v_1, v_2, v_3, \ldots, v_p$ be the vertices of $W_p$ such that $\deg(v_i) = 3$ for $1 \leq i \leq p - 1$ and $\deg(v_p) = p - 1$. Let $e_1, e_2, e_3, \ldots, e_{p-1}$, $e'_1, e'_2, e'_3, \ldots, e'_{p-1}$ be the edges of $W_p$ such that $e_i = v_iv_{i+1}$ for $1 \leq i \leq p - 2$, $e_{p-1} = v_pv_{p-1}$ and $e'_i = v'_iv'_p$ for $1 \leq i \leq p - 1$. Now, in $L(W_p)$ $e_1, e_2, e_3, \ldots, e_{p-1}, e'_1, e'_2, e'_3, \ldots, e'_{p-1}$ be the vertices of $L(W_p)$ which corresponds to the edges of $W_p$. In $L(T)$, $F_1 = \{ e_1, e_2, e_3, \ldots, e_{p-2}, e_{p-1}, e'_1, e'_2, e'_3, \ldots, e'_{p-2}, e'_{p-1} \}$ in which $\deg(e_i) = 4 = \deg(e'_i) \forall e_i, e'_i \in F_1$ and $< F_1 >$ is 4-regular. Similarly for $F_2 = \{ e_1, e'_2, e'_3, \ldots, e'_1, e_{p-2}, e_{p-1} \} \cup e'_k \in F_2$, $\deg(e'_k) = p - 4, k = 1, 2, 3, \ldots, p - 1$, is the another partition such that $< F_2 >$ is $(p - 4)$ regular graph. Thus, $r_l(W_p) = \frac{1}{2} |F_1, F_2| = 2$.

In the next result we establish the relationship between $r_l(T)$ and $n$ where $n$ is n-distinct cut vertices.

**Theorem 8:** For any non-trivial tree $T$, with n-distinct cut vertices, $r_l(T) = n$.

**Proof:** Suppose $G = T$ has unique cut vertex incident with $m$ number of edges in $L(G)$, the sub graph $L(G) = K_m$ which is $m$-regular. Now, $r_l(T) = n = 1$. Further, assume $n \geq 2$ and each $n$ is distinct. Let $T$ has $\{2\}, \{3\}, \ldots, \{n\}$, number of cut vertices and each set has $v_1, v_2, v_3, \ldots, v_n$ be the number of cut vertices in $T$; $\deg(v_1) \neq \deg(v_2) \neq \deg(v_3) \neq \ldots \neq \deg(v_n)$ and $\forall v_i \in T$ has degree at least 2. Since each $v_i, 1 \leq i \leq n$ are distinct. Let $\{ e_1, e_2, e_3, \ldots, e_p \}$ be the number of edges in $T$, which are incident to $\forall v_i 1 \leq i \leq n$. Then in $L(T)$, the number of edges incident to each $v_i \in G$ generates a complete block in $L(T)$. In $L(T)$ each block is of different regular and is complete. Since $T$ has n-distinct cut vertices and each block is of different regular, then each block belongs to different partition of $L(T)$ such that there exists $F_1, F_2, F_3, \ldots, F_n$ partitions of $L(T)$.

Hence, $r_l(T) = \frac{1}{n} \{ F_1, F_2, \ldots, F_n \}$ which gives $r_l(T) = n$.

Next, we developed the result which gives the relationship between $r_l(G)$ and $diam[L(G)]$.

**Theorem 9:** For any graph $G$, $r_l(G) \leq q - diam[L(G)] + 1$.

**Proof:** Let $P_n : \{ v_1v_2, v_2v_3, v_3v_4, \ldots, v_{p-2}v_{p-1}, v_{p-1}v_p \}$ be a path on $diam[L(G)] = 2$ vertices. Let $e_1 = v_1v_2, e_2 = v_2v_3, e_3 = v_3v_4, e_4 = v_4v_5, \ldots,$
Hence T has n distinct degree cut vertices gives
∈ (L(P)) such that cut vertices, then between
In the following theorem we establish the relationship
Proof: Theorem 10:
Suppose every vertex of D is adjacent to at least one end vertex. Then
D is a γ-set of T. Suppose deg(v_1) > deg(v_2) > deg(v_3) > \ldots > deg(v_n). Then in L(G) ∪ v_i ∈ D gives n-distinct
regular partition. Hence |D| = n = γ(T). For inequality suppose D_1 = {v_1, v_2, v_3, \ldots, v_k} such that D_1 ⊂ D and ∀v_i ∈ D_1 are not in γ-set of T. We consider the following cases.
Case 1. Assume v_i ∈ D_1 are not adjacent to each other. Then, in L(T) the edges incident to each v_i forms a regular block in L(T) which are not adjacent. Hence there blocks which are with same regular and are not adjacent to each other given only one partition. Remaining blocks in L(T) are in different regular classes. Hence, r_1(T) ≤ γ(T).
Case 2. Assume v_i ∈ D_1 are adjacent. Then in L(T) the edges incident to these vertices form same regular and hence they are in different regular partition. Thus |D_1| > γ(T) which gives r_1(T) > γ(T). Hence in all these we have r_1(T) ≤ γ(T).
3 Conclusion
We established the regular number of line graph of some standard graphs by replacing the each edge by a vertex. Also many results established are sharp.
4 References