Isomorphism and Antiisomorphism in (S, Q)-Fuzzy Translation of (S, Q)-Fuzzy Subhemirings of a Hemiring

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Abstract
In this paper, we made an attempt to study the algebraic nature of a(S, Q)-fuzzy subhemiring of a Hemiring.
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Introduction
There are many concepts of universal algebras generalizing an associative ring (R, +, ·). Some of them in particular, near rings and several kinds of semirings have been proven very useful. Semirings (called also half rings) are algebras (R; +, ·) share the same properties as a ring except that (R; +) is assumed to be a semi group rather than a commutative group. Semi rings appear in a natural manner in some applications to the theory of automata and formal languages. An algebra (R; +, ·) is said to be a semi ring (R; +) and (R; ·) are semi groups satisfying a.(b+c)=a.b+a.c and (b+c).a=b.a+c.a for all a, b and c in R. A semiring R is said to be additively commutative if a+ b = b+ a for all a, b and c in R. A semiring R may have an identity 1, defined by 1+a=a=1 and a zero 0, defined by 0+a=a=0 and a.0=0=0.a for all a in R. A semiring R is said to be a hemi ring if it is an additively commutative with zero. After the introduction of fuzzy sets by L.A. Zadeh \cite{16}, several researchers explores on the generalization of the concept of fuzzy sets. Osman Kazanci, Sultan yamark and serifeyilmaz in \cite{11} have introduced the Notion of intuitionistic Q-fuzzification of N-subgroups (subnear rings) in a near-ring and investigated some related properties. Solairaju. A and R. Nagarajan, have given a new structure in construction of Q-fuzzy groups and subgroups \cite{14, 15}. In this paper, we introduce some properties and theorems in (S,Q)-fuzzy subhemirings of a hemiring.

1. Preliminaries
1.1 Definition: A S-norm is a binary operation \( S: [0,1] \times [0,1] \to [0,1] \) satisfying the following requirements:
(i) \( S(x, 1) = x \) for all \( x \in [0,1] \) (boundary conditions)
(ii) \( S(x, y) = S(y, x) \) (symmetry)
(iii) \( S(x, y) \leq S(x, z) \) for all \( x, y, z \in [0,1] \) (monotonicity)

1.2 Definition: Let X be a non-empty set and Q be a non-empty set. A Q-fuzzy subset A of X is defined by \( A: X \to [0,1] \).

1.3 Definition: The union of two (S,Q)-fuzzy sets A and B is defined by \( (A \cup B)(x, q) = \max \{ S(A(x, q)), S(B(x, q)) \} \) for all \( x \in X \) and q in Q.

1.4 Definition: The intersection of two (S,Q)-fuzzy sets A and B is defined by \( (A \cap B)(x, q) = \min \{ S(A(x, q)), S(B(x, q)) \} \) for all \( x \in X \) and q in Q.
1.5 Definition: Let $(R, +, \cdot)$ be a hemiring. A $(S, Q)$-fuzzy subset $A$ of $R$ is said to be a $(S, Q)$-fuzzy subhemiring (SQFSHR) of $R$ if it satisfies the following conditions:
(i) $\mu_A(x + y, q) \in S(\mu_A(x, q), \mu_A(y, q))$
(ii) $\mu_A(xy, q) \in S(\mu_A(x, q), \mu_A(y, q))$, for all $x$ and $y$ in $R$, and $q$ in $Q$.

1.6 Definition: Let $(R, +, \cdot)$ be a hemiring. A $(S, Q)$-fuzzy subhemiring $A$ of $R$ is said to be a $(S, Q)$-fuzzy normal subhemiring (SQFNSHR) of $R$ if
$S(\mu_A(xy, q)) = S(\mu_A(yx, q))$, for all $x$ and $y$ in $R$, and $q$ in $Q$.

1.7 Definition: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Then the function $f: R \rightarrow R'$ is called a hemiring homomorphism if it satisfies the following axioms:
i) $f(x + y) = f(x) + f(y)$,
ii) $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$.

1.8 Definition: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Then the function $f: R \rightarrow R'$ is called a hemiring anti-homomorphism if it satisfies the following axioms:
i) $f(x + y) = f(y) + f(x)$,
ii) $f(xy) = f(y)f(x)$, for all $x$ and $y$ in $R$.

1.9 Definition: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Then the function $f: R \rightarrow R'$ be a hemiring homomorphism. If $f$ is one-to-one and onto, then $f$ is called a hemiring isomorphism.

1.10 Definition: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Then the function $f: R \rightarrow R'$ be a hemiring anti-homomorphism. If $f$ is one-to-one and onto, then $f$ is called a hemiring anti-isomorphism.

1.11 Definition: Let $A$ be a $(S, Q)$-fuzzy subset of $X$ and $\alpha \in [0, 1] - \text{Sup}\left\{ A(x, q) : x \in X, 0 < A(x, q) < 1 \right\}$. Then $T = T^\alpha_A$ is called a $(S, Q)$-fuzzy translation of $A$ if $S(\mu_T(x, q)) = S(\mu_A(x, q) + \alpha)$, for all $x$ in $X$.

2. Isomorphism and Antiisomorphism in (S, Q)-Fuzzy Translation of (S, Q)-Fuzzy Subhemirings of a Hemiring

2.1 Theorem: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. The $(S, Q)$-fuzzy normal subhemiring $V$ of $(R') = \text{antihomomorphic preimage}$ is a $(S, Q)$-fuzzy normal subhemiring of $R$.

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Let $f: R \rightarrow R'$ be an anti-homomorphic preimage of $V$. Then we have to prove that $A$ is a $(S, Q)$-fuzzy normal subhemiring of hemiring $R$. Let $x$ and $y$ in $R$ and $q$ in $Q$. Then clearly $A$ is a $(S, Q)$-fuzzy subhemiring of the hemiring $R$. Since $V$ is a $(S, Q)$-fuzzy subhemiring of the hemiring $R'$, we have to prove that $A$ is a $(S, Q)$-fuzzy normal subhemiring of hemiring $R$. Let $x$ and $y$ in $R$ and $q$ in $Q$. Then clearly $A$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$. Now, $S(\mu_A(xy, q)) = S(\mu_V((f(xy), q)))$, since $S(\mu_A(x, q)) = S(\mu_V((fx, q))) = S(\mu_V((fy, f(x), q)))$ as $f$ is an anti-homomorphism = $S(\mu_V((fy, f(x), q)))$ as $f$ is an anti-homomorphism = $S(\mu_V((fy, f(x), q)))$ as $f$ is an anti-homomorphism = $S(\mu_A((xy, q)))$, which implies that $S(\mu_A((xy, q))) = S(\mu_A((yx, q)))$ for all $x$ and $y$ in $R$, and $q$ in $Q$. Hence $A$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$.

In the following Theorem $\circ$ is the composition operation of functions:

2.2 Theorem: Let $A$ be a $(S, Q)$-fuzzy subhemiring of the hemiring $H$ and $f$ is an isomorphism from a hemiring $R$ onto $H$. If $A$ be a $(S, Q)$-fuzzy normal subhemiring of the hemiring $H$, then $A \circ f$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$.

Proof: Let $x$ and $y$ in $R$ and $q$ in $Q$ and $A$ be a $(S, Q)$ fuzzy normal subhemiring of the hemiring $H$. Then we have, Clearly $A \circ f$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$. Now, $S(\mu_A((f(xy), q))) = S(\mu_A(\circ f)(xy, q))$ as $f$ is an isomorphism = $S(\mu_A(f(y)f(x), q))$ as $f$ is an isomorphism = $S(\mu_A((yx, q)))$ for all $x$ and $y$ in $R$, and $q$ in $Q$. Therefore $A \circ f$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$.

2.3 Theorem: Let $A$ be a $(S, Q)$-fuzzy subhemiring of the hemiring $H$ and $f$ is an anti-isomorphism from a hemiring $R$ onto $H$. If $A$ be a $(S, Q)$-fuzzy normal subhemiring of the hemiring $H$, then $A \circ f$ is a $(S, Q)$-fuzzy normal subhemiring of the hemiring $R$.

Proof: Let $x$ and $y$ in $R$ and $q$ in $Q$ and $A$ be a $(S, Q)$ fuzzy normal subhemiring of the hemiring $H$. Then we have, Clearly $A \circ f$ is a $(S, Q)$-fuzzy subhemiring of the hemiring $R$. Now, $S(\mu_A(\circ f)(xy, q)) = S(\mu_A(f(xy), q))$ as $f$ is an anti-isomorphism = $S(\mu_A(f(y)f(x), q))$ as $f$ is an anti-isomorphism = $S(\mu_A((yx, q)))$, which implies that $S(\mu_A((xy, q))) = S(\mu_A(f)((xy, q)))$, for all $x$ and $y$ in $R$ and $q$ in $Q$. Therefore $A \circ f$ is a $(S, Q)$ fuzzy normal subhemiring of the hemiring $R$.

2.4 Theorem: If $M$ and $N$ are two $(S, Q)$-fuzzy translations of $(S, Q)$-fuzzy normal subhemiring $A$ of a hemiring $(R, +, \cdot)$, then their intersection $M \cap N$ is a $(S, Q)$-fuzzy translation of $A$. 
Proof: It is trivial.

2.5 Theorem: The intersection of family of $(S, Q)$-fuzzy translations of $(S, Q)$ fuzzy normal subhemiring $A$ of a hemiring $(R, +,.)$ is a $(S,Q)$-fuzzy translation of $A$.

Proof: It is trivial.

2.6 Theorem: If $M$ and $N$ are two $(S,Q)$-fuzzy translations of $(S, Q)$ fuzzy normal subhemiring $A$ of a hemiring $(R, +,.)$, then their union $M \cup N$ is a $(S,Q)$-fuzzy translation of $A$.

Proof: It is trivial.

2.7 Theorem: The union of family of $(S, Q)$-fuzzy translations of $(S, Q)$ fuzzy normal subhemiring $A$ of a hemiring $(R, +,.)$ is a $(S,Q)$-fuzzy translation of $A$.

Proof: It is trivial.

2.8 Theorem: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set. If $f: R \to R'$ is a homomorphism, then $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring $A$ of $R$ under the homomorphic image is $(S,Q)$-fuzzy normal subhemiring of $f(R) = R'$.

Proof: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set and $f: R \to R'$ be a homomorphism. That is $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $T = T_A^V$ be the $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring of $V$ of $R'$ and $A$ be the homomorphic image of $T$ under $f$. We have to prove that Visa $(S,Q)$ fuzzy normal subhemiring of $R'$. Now, $S(V(f(x)y), q) = S(V(f(x)f(y), q)) \geq S(T(xy), q) = S(A(xy, q) + \alpha) = S(A(xy, q) + \alpha) = S(T(xy), q)$, which implies that $S(V(f(x)y), q) \geq S(V(f(x)f(y), q))$ for all $f(x)$ and $f(y)$ in $R'$ and $q$ in $Q$. Therefore $V$ is a $(S,Q)$-fuzzy normal subhemiring of the hemiring $R'$.

2.9 Theorem: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set. If $f: R \to R'$ is a homomorphism, then $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring $V$ of $(R) = R'$ under the homomorphic pre-image is $(S,Q)$-fuzzy normal subhemiring of $R$.

Proof: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set and $f: R \to R'$ be a homomorphism. That is $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $T = T_A^V$ be the $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring of $V$ of $R'$ and $A$ be the homomorphic image of $T$ under $f$. We have to prove that Visa $(S,Q)$-fuzzy normal subhemiring of $R'$. Now, $S(A(xy, q)) = S(T(f(xy), q)) = S(V(f(xy), q)) \geq S(T(xy), q) = S(A(xy, q) + \alpha) = S(A(xy, q) + \alpha) = S(T(xy), q)$, which implies that $S(A(xy, q)) \geq S(V(f(xy), q))$ for all $x$ and $y$ in $R$ and $q$ in $Q$. Therefore $A$ is a $(S,Q)$-fuzzy normal subhemiring of the hemiring $R$.

2.10 Theorem: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set. If $f: R \to R'$ is an anti-homomorphism, then $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring $A$ of $R$ under the anti-homomorphic image is $(S,Q)$-fuzzy normal subhemiring of $f(R) = R'$.

Proof: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set and $f: R \to R'$ be an anti-homomorphism. That is $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all $x$ and $y$ in $R$. Let $T = T_A^V$ be the $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring of $A$ of $R$ and $V$ be the anti- homomorphic image of $T$ under $f$. We have to prove that Visa $(S,Q)$-fuzzy normal subhemiring of $R'$. Now, $S(V(f(x)y), q) = S(V(f(x)f(y), q)) \leq S(T(xy), q) = S(A(xy, q) + \alpha) = S(A(xy, q) + \alpha) = S(T(xy), q)$, which implies that $S(V(f(x)y), q) \leq S(V(f(x)f(y), q))$, for all $f(x)$ and $f(y)$ in $R'$ and $q$ in $Q$. Therefore $V$ is a $(S,Q)$-fuzzy normal subhemiring of the hemiring $R'$.

2.11 Theorem: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set. If $f: R \to R'$ is an anti-homomorphism, then $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring $V$ of $(R) = R'$ under the anti-homomorphic pre-image is $(S,Q)$-fuzzy normal subhemiring of $R$.

Proof: Let $(R, +,.)$ and $(R', +,.)$ be any two hemirings and $Q$ be a non-empty set and $f: R \to R'$ be an anti-homomorphism. That is $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all $x$ and $y$ in $R$. Let $T = T_A^V$ be the $(S,Q)$-fuzzy translation of a $(S,Q)$-fuzzy normal subhemiring of $V$ of $R'$ and $A$ be the anti-homomorphic pre-image of $T$ under $f$. We have to prove that Visa $(S,Q)$-fuzzy normal subhemiring of $R'$. Let $x$ and $y$ in $R$ and $q$ in $Q$. Then clearly, $V$ is a $(S,Q)$-fuzzy normal subhemiring of the hemiring $R$. Therefore $V$ is a $(S,Q)$-fuzzy normal subhemiring of the hemiring $R'$.

It is trivial.
Now, $S(A(xy,q)) = S(T(f(xy),q)) = S(V(f(xy),q) + \alpha) = S(V(f(y)f(x),q) + \alpha) = S(V(f(x)f(y),q) + \alpha) = S(T(f(yx),q)) = S(A(yx,q))$, which implies that $S(A(xy,q)) = S(A(yx,q))$, for all $x$ and $y$ in $R$ and $q$ in $Q$. Therefore, $A$ is a $(S,Q)$-fuzzy normal subhemiring of $R$.

**Reference**