Anti (S, Q)-fuzzy Subhemirings of a Hemiring

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Abstract
In this paper, an attempt has been made to study the algebraic nature of an anti(S, Q)-fuzzy subhemiring of a hemiring.

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Introduction
There are many concepts of universal algebras generalizing an associative ring \((R; +; .)\). Some of them, in particular, about near rings and several kinds of semirings have been proved very useful. Semirings (also called half rings) are algebras \((R; +; .)\) which share the same properties as a ring excepting that \((R; +)\) is assumed to be a semi group rather than a commutative group. Semi rings appear in a natural manner in some applications the theory of automata and formal languages. An algebra \((R; +; .)\) is said to be a semi ring \((R; +)\) and \((R; .)\) are semi groups satisfying \(a.(b+c)=a.b+a.c\) and \((b+c).a=b.a+c.a\) for all \(a, b\) and \(c\) in \(R\). A semiring \(R\) is said to be additively commutative if \(a+b=b+a\) for all \(a, b\) and \(c\) in \(R\). A semiring \(R\) may have an identity 1, defined by \(1.a=a=a.1\) and a zero 0, defined by \(0+a=a=a+0\) and \(a.0=0=0.a\) for all \(a\) in \(R\). A semiring \(R\) is said to be a hemi ring if it is additively commutative with zero. After the introduction of fuzzy sets by L.A. Zadeh [12], several researchers explored the generalization of the concept of fuzzy sets. The notion of anti-left h-ideals in hemi ring was introduced by Akram. M and K.H. Dar [1]. The notion of homomorphism and anti-homomorphism of fuzzy and anti-fuzzy ideal of a ring was introduced by N. Palaniappan & K. Arjunan [6]. Osman Kanzanci, Sultan Yamark and Serife Yilmaz in [13] introduced the notion of intuitionistic Q-fuzzification of N-subgroups (sub near-rings) in a near –ring and investigated some related properties. A. Solairaju and R. Nagarajan have given a new structure in construction of Q-fuzzy groups and subgroups [14][15]. This paper introduces some properties and theorems in \((S,Q)\)-fuzzy subhemirings of a hemiring.

1. Preliminaries
1.1 Definition: A S-norm is a binary operation \(S: [0,1] \times [0,1] \rightarrow [0,1]\) satisfying the following requirements:
   (i) \(0 \leq S(x, y) \leq 1\) (boundary conditions)
   (ii) \(x \leq y \Rightarrow S(x, y) = S(y, y)\) (commutativity)
   (iii) \(x \leq (y \leq z) \Rightarrow S(x, y) \leq S(y, z)\) (associativity)
   (iv) \(S(0, x) = S(x, 0)\) (monotonicity).

1.2 Definition: Let \(X\) be a non-empty set and \(Q\) be a non-empty set. A \(Q\)-fuzzy subset \(A\) of \(X\) is function \(A: X \times Q \rightarrow [0,1]\).

1.3 Definition: Let \((R, +, .)\) be a hemiring. A \(Q\)-fuzzy subset of \(R\) is said to be an anti \((S,Q)\)-fuzzy subhemiring(anti \(Q\)-fuzzy subhemiring with respect to \(S\)-norm) of \(R\) if it satisfies the following conditions:
   (i) \(\mu_A(x + y, q) \leq S(\mu_A(x, q), \mu_A(y, q))\)
   (ii) \(\mu_A(xy, q) \leq S(\mu_A(x, q), \mu_A(y, q))\) for all \(x\) and \(y\) in \(R\), and \(q\) in \(Q\).
1.4 Definition: Let A and B be (S, Q)-fuzzy subsets of sets G and Respectively. The anti-product of A and B, denoted as AB, is defined as $AB = \left\{ (p, q) \mid \mu_A(x, p) \land \mu_B(y, q) \right\}$ for all x in G and y in H & q in Q, where

$$\mu_{AB}(x, y, q) = \max\{ \mu_A(x, q), \mu_B(y, q) \}.$$

1.5 Definition: Let A be a Q-fuzzy subset in a set S, the anti-strongest relation Q-fuzzy relation on S, that is a Q-fuzzy relation on A is V given by $\mu_V((x, y), q) = \max\{ \mu_A(x, q), \mu_B(y, q) \}$ for all x and y in S and q in Q.

1.6 Definition: Let (R, +, .) and (R', +, .) be any two hemirings. Let f: R → R' be any function and A and B be an anti (S, Q)-fuzzy subhemiring in R, V be an anti (S, Q)-fuzzy subhemiring in f(R) = R', defined by $\mu_V(y, q) = \inf_{x \in R} \mu_A(x, q)$ for all x in R and y in R' and q in Q. Then A is called a preimage of V under f and is denoted by $f^{-1}(V)$.

1.7 Definition: Let A be an anti(S, Q)-fuzzy subhemiring of a hemiring (R, +, .) and a in R, then the pseudo anti (S, Q)-fuzzy coset (aA)$^p$ is defined by $(aA)^p(x, q) = p^{(a)}S(\mu_A(x, q), q)$, for every x in R, q in Q and for some p in P.

2. Properties Of Anti (S, Q)-Fuzzy Subhemiring Of A Hemiring

2.1 Theorem: Union of any two anti (S, Q)-fuzzy subhemirings of a hemiring R is an anti (S, Q)-fuzzy subhemiring of R.

Proof: Let A and B be any anti (S, Q)-fuzzy subhemirings of a hemiring R and x and y in R. Let $A = \{(x, q) \mid \mu_A(x, q) \geq \mu(x, q) \}$ and $B = \{(x, q) \mid \mu_B(x, q) \geq \mu(x, q) \}$ for all x in R and q in Q. Therefore, $A \cup B = \max\{\mu_A(x, q), \mu_B(x, q)\} \geq \mu(x, q)$ for all x in R and q in Q.

2.2 Theorem: The Union of a family of anti(S, Q)-fuzzy subhemirings of hemiring R is an anti (S, Q)-fuzzy subhemiring of R.

Proof: Let A and B be any anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Then $A \cup B$ is an anti (S, Q)-fuzzy subhemiring of $(R \times R)$.

2.3 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R\times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.

2.4 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R \times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.

2.5 Theorem: If A is an anti (S, Q)-fuzzy subhemiring of a hemiring (R, +, .) if and only if $\mu_A(x + y, q) \leq S(\mu_A(x, q), \mu_B(y, q))$ for all x and y in R. The anti-product of A and B is defined as $AB = \{(x, y) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y) \in R \times R$.

2.6 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R \times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.

2.7 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R \times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.

2.8 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R \times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.

2.9 Theorem: If A and B are two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$, respectively, then anti product $A \times B$ is an anti (S, Q)-fuzzy subhemiring of $R \times R \times R$.

Proof: Let A and B be two anti(S, Q)-fuzzy subhemirings of the hemirings $R \times R$. Let $x \in R, y \in R$, and $x', y' \in R$. Then $(x, y) \in A \times B$ if and only if $x \in A$ and $y \in B$. Therefore, $A \times B = \{(x, y, q) \mid \mu_A(x, q) \land \mu_B(y, q) \} \geq \mu_A(x, q) \land \mu_B(y, q)$ for all $(x, y, q) \in R \times R$.
2.6 **Theorem:** If \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of a hemiring \((R, +, .)\), then \( H = \{ x/x \in R : \mu_A(x, q) = 0 \} \) is either empty or is a subhemiring of \( R \).

**Proof:** It is trivial.

2.7 **Theorem:** Let \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of a hemiring \((R, +, .)\).

\[ \exists \mu_A(x + y, q) = 1 \] then either \( \mu_A(x, q) = 1 \) or \( \mu_A(y, q) = 1 \), for all \( x \) and \( y \) in \( R \).

**Proof:** It is trivial.

2.8 **Theorem:** Let \( A \) is an anti \((S,Q)\)-fuzzy subhemiring of a hemiring \((R, +, .)\), then the pseudo anti \((S,Q)\)-fuzzy coset \( (aA) \) is an anti \((S,Q)\)-fuzzy subhemiring of a hemiring \( R \), for every \( a \) in \( R \).

**Proof:** Let \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of a hemiring \( R \). For every \( x \) and \( y \) in \( R \), we have \( ((a \mu_A)^p(x + y, q) \leq p(s)S((a \mu_A(x, q), p(s)\mu_A(y, q)) \in S(\mu_A(x, q), \mu_A(y, q)) \in S(a \mu_A(x, q), a \mu_A(y, q)) \).

Now, \((a \mu_A)^p(x, q) \leq \mu_A(x, q) \leq \mu_A(y, q) \in S(\mu_A(x, q), \mu_A(y, q)) \). Therefore, \((a \mu_A)^p(x, q) \leq \mu_A(y, q) \in S(\mu_A(x, q), \mu_A(y, q)) \).

\[ \mu_A((xy, q) \leq S(\mu_A(x, q), \mu_A(y, q)) \in S(\mu_A(x, q), \mu_A(y, q)) \]. Hence \((aA)^p \) is an anti \((S, Q)\)-fuzzy subhemiring of a hemiring \( R \).

2.9 **Theorem:** Let \((R, +, .) \) and \((R, +, .) \) be any two hemirings. The homomorphic image of an anti \((S, Q)\)-fuzzy subhemiring of \( R \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \).

**Proof:** Let \( f : R \rightarrow R' \) be a homomorphism. Then, \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \) in \( R \). Let \( V = f(A) \), where \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \). Now, for \( f(x), f(y) \) in \( R' \), \( \mu_V((f(x) + (f(y)), q) \leq \mu_A(x + y, q) \leq S(\mu_A(x, q), \mu_A(y, q)) \). which implies that \( \mu_V((f(x) + (f(y)), q) \leq S(\mu_A(x, q), \mu_A(y, q)) \).

\[ \mu_V((f(xy), q) \leq S(\mu_A(x, q), \mu_A(y, q)) \in S(\mu_A(x, q), \mu_A(y, q)) \]. Hence \( V \) is an anti \((S, Q)\)-fuzzy subhemiring of hemiring \( R' \).

2.10 **Theorem:** Let \((R, +, .) \) and \((R, +, .) \) be any two hemirings. The homomorphic preimage of an anti \((S, Q)\)-fuzzy subhemiring of \( R \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \).

**Proof:** Let \( f : R \rightarrow R' \) be a homomorphism. Then, \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \) in \( R \). Let \( V = f(A) \), where \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R' \).

Now, for \( x, y \) in \( R \), \( \mu_A(x + y, q) = \mu_V((f(x) + (f(y)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \in S(\mu_A(x, q), \mu_A(y, q)) \). which implies that \( \mu_V((f(x) + (f(y)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \).

\[ \mu_V((f(xy), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \in S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \]. Hence \( V \) is an anti \((S, Q)\)-fuzzy subhemiring of hemiring \( R \).

2.11 **Theorem:** Let \((R, +, .) \) and \((R, +, .) \) be any two hemirings. The anti-homomorphic image of an anti \((S, Q)\)-fuzzy subhemiring of \( R \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \).

**Proof:** Let \( f : R \rightarrow R' \) be a homomorphism. Then, \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \) in \( R \). Let \( V = f(A) \), where \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R' \).

Now, for \( x, y \) in \( R \), \( \mu_A(x + y, q) = \mu_V((f(x) + (f(y)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \in S(\mu_A(x, q), \mu_A(y, q)) \). which implies that \( \mu_V((f(x) + (f(y)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \).

\[ \mu_V((f(xy), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \in S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \]. Hence \( V \) is an anti \((S, Q)\)-fuzzy subhemiring of hemiring \( R \).

2.12 **Theorem:** Let \((R, +, .) \) and \((R, +, .) \) be any two hemirings. The anti-homomorphic preimage of an anti \((S, Q)\)-fuzzy subhemiring of \( R \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \).

**Proof:** Let \( V = f(A) \) where \( V \) is an anti \((S, Q)\)-fuzzy subhemiring of \( R \). Let \( x \) and \( y \) in \( R \). Then \( \mu_A((x + y, q) = \mu_V((f(x) + (f(y)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \in S(\mu_A(x, q), \mu_A(y, q)) \). which implies that \( \mu_A((x + y, q) \leq S(\mu_A(x, q), \mu_A(y, q)) \). Again \( \mu_A((xy, q) = \mu_V((f(xy)), q) \leq S(\mu_V((f(x)), q), \mu_V((f(y)), q)) \). which implies that \( \mu_A((xy, q) \leq S(\mu_A(x, q), \mu_A(y, q)) \). Hence \( A \) is an anti \((S, Q)\)-fuzzy subhemiring of hemiring \( R \).
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2.13 Theorem: Let A be an anti (S, Q)-fuzzy subhemiring of hemiring H and f is an isomorphism from a hemiring R onto H. Then \( A \circ f \) is an anti (S, Q)-fuzzy subhemiring of R.

Proof: Let \( x \) and \( y \) in R. Then we have, 
\[
(\mu \circ f)(x + y, q) = \mu_A((f(x), q) + (f(y), q)) \leq S(\mu_A(f(x), q), \mu_A(f(y), q)) \leq S((\mu \circ f)(x, q), (\mu \circ f)(y, q)),
\]
which implies that 
\[
(\mu \circ f)((x + y, q)) \leq S(\mu_A(f(x), q), (\mu \circ f)(y, q)).
\]
And, 
\[
(\mu \circ f)((xy, q)) = \mu_A((f(x), q)(f(y), q)) \leq S((\mu \circ f)(x, q), \mu_A(f(y), q)),
\]
which implies that 
\[
(\mu \circ f)((xy, q)) \leq S((\mu \circ f)(x, q), (\mu \circ f)(y, q)).
\]
Therefore \( A \circ f \) is an anti (S, Q) fuzzy subhemiring of hemiring R.

2.14 Theorem: Let A be an anti (S, Q)-fuzzy subhemiring of hemiring H and f is an anti-isomorphism from a hemiring R onto H. Then \( A \circ f \) is an anti (S, Q)-fuzzy subhemiring of R.

Proof: Let \( x \) and \( y \) in R. Then we have, 
\[
(\mu \circ f)((x + y, q)) = \mu_A((f(x), q) + (f(y), q)) \leq S(\mu_A(f(x), q), (\mu \circ f)(y, q)) \leq S((\mu \circ f)(x, q), (\mu \circ f)(y, q)),
\]
which implies that 
\[
(\mu \circ f)((x + y, q)) \leq S((\mu \circ f)(x, q), (\mu \circ f)(y, q)).
\]
And, 
\[
(\mu \circ f)((xy, q)) = \mu_A((f(x), q)(f(y), q)) \leq S((\mu \circ f)(x, q), \mu_A(f(y), q)),
\]
which implies that 
\[
(\mu \circ f)((xy, q)) \leq S((\mu \circ f)(x, q), (\mu \circ f)(y, q)).
\]
Therefore \( A \circ f \) is an anti (S, Q) fuzzy subhemiring of hemiring R.

Reference