On $W_{Ig}$-Continuous and $W_{Ig^*}$-Continuous Functions in Ideal Topological Spaces

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Abstract

In this paper we introduce and study the notions of $W_{Ig}$-continuous and $W_{Ig^*}$-continuous, $W_{Ig}$-irresolute and $W_{Ig^*}$-irresolute in ideal topological spaces, and also we studied their properties.

Keywords: $W_{Ig}$-closed, $W_{Ig^*}$-closed, $W_{Ig}$-continuous, $W_{Ig^*}$-continuous, $W_{Ig}$-irresolute, $W_{Ig^*}$-irresolute.

Introduction

Ideals in topological spaces have been considered since 1930. In 1990, Jankovic and Hamlett [5] once again investigated applications of topological ideals. The notion of $I_{g}$-closed sets was first by Dontchev.et.al [3] in 1999. Navaneethakrishnan and Joseph [9] further investigated and characterized $I_{g}$-closed sets and $I_{g}$-open sets by the use of local functions. The notion of $I_{g^*}$-closed sets was introduced by Ravi.et.al [10] in 2013. Recently the notion of $W_{Ig}$-closed sets and $W_{Ig^*}$-closed sets was introduced and investigated by Maragathavalli.et.al [8]. In this paper, we introduce the notions of $W_{Ig}$-continuous and $W_{Ig^*}$-continuous functions in ideal topological spaces.

An ideal I on a topological space $(X, \tau)$ is a non-empty collection of subsets of X which satisfies the following properties. (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space $(X, \tau, I)$ with an ideal I on X and is denoted by $(X, \tau, I)$. For a subset $A \subseteq X$, $A^{*}(I, \tau) = \{ x \in X : A \cap U \notin I \text{ for every } U \in \tau(X, x) \}$ is called the local function of A with respect to I and $\tau$ [6]. We simply write $A^{*}$ in case there is no chance for confusion. A Kuratowski closure operator $cl^{*}(.)$ for a topology $\tau^{*}(I, \tau)$ called the *-topology, finer than $\tau$ is defined $cl^{*}(A) = A \cup A^{*}$ [11]. If $A \subseteq X$, $cl(A)$ and $int(A)$ will respectively, denote the closure and interior of A in $(X, \tau)$.

Definition 1.1. A subset A of a topological space $(X, \tau)$ is called

1. $g$-closed [7], if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X, \tau)$.
2. $\tilde{g}$-closed [12], if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in $(X, \tau)$.

Definition 1.2. A subset A of a topological space $(X, \tau, I)$ is called

1. $I_{g}$-closed [8], if $A^{*} \subseteq U$ whenever $A \subseteq U$ and U is open in X.
2. $I_{g}$-closed [11], if $A^{*} \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
3. $W_{Ig}$-closed [8], if $int(A^{*}) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
4. $W_{Ig^*}$-closed [8], if $int(A^{*}) \subseteq U$ whenever $A \subseteq U$ and U is $\tilde{g}$-open in X.
5. $g^{*}$-closed [10], if $A^{*} \subseteq U$ whenever $A \subseteq U$ and U is $\tilde{g}$-open in $(X, \tau)$.

Definition 1.3. A function $f: (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be

1. $g$-continuous [2], if for every open set $V \in \sigma$, $f^{-1}(V)$ is $g$-open in $(X, \tau)$.
2. $\tilde{g}$-continuous [12], if for every open set $V \in \sigma$, $f^{-1}(V)$ is $\tilde{g}$-open in $(X, \tau)$.

Definition 1.4. A function $f: (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be $I_{g}$-continuous [5], if $f^{-1}(V)$ is $I_{g}$-closed in $(X, \tau, I)$ for every closed set V in $(Y, \sigma)$.
2. $\text{wl}_g$-continuous and $\text{wl}_{g^*}$-continuous.

**Definition 2.1:** A function $f : (X, \tau, I) \to (Y, \sigma)$ is Said to be 1. Weakly $I_g$-continuous (briefly $\text{wl}_g$-continuous) if $f^{-1}(V)$ is weakly $I_g$-closed in $(X, \tau, I)$ for every closed set $V$ in $(Y, \sigma)$.

2. Weakly $I_{g^*}$-continuous (briefly $\text{wl}_{g^*}$-continuous) if $f^{-1}(V)$ is weakly $I_{g^*}$-closed in $(X, \tau, I)$ for every closed set $V$ in $(Y, \sigma)$.

**Definition 2.2:** A function $f : (X, \tau, I_1) \to (Y, \sigma, I_2)$ is Said to be (i) $\text{wl}_g$-irresolute if $f^{-1}(V)$ is $\text{wl}_g$-closed in $(X, \tau, I_1)$ for every $\text{wl}_g$-closed set $V$ in $(Y, \sigma, I_2)$.

(ii) $\text{wl}_{g^*}$-irresolute if $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed in $(X, \tau, I_1)$ for every $\text{wl}_{g^*}$-closed set $V$ in $(Y, \sigma, I_2)$.

**Theorem 2.3:** Ever continuous function is $\text{wl}_g$-continuous.

**Proof:** Let $f$ be a continuous function and let $V$ be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is closed set in $(X, \tau, I)$. Since every closed set is $\text{wl}_g$-closed. Hence $f^{-1}(V)$ is $\text{wl}_g$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_g$-continuous.

**Example 2.4:** Let $X = Y = \{a, b, c\}$, $\tau = \{\varphi, \{b\}, \{b, c\}, X\}$, $\sigma = \{\varphi, \{c\}, \{a, c\}, Y\}$ and $I = \{\varphi, \{c\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_g$-continuous but not $\text{wl}_{g^*}$-continuous.

**Theorem 2.5:** Ever continuous function is $\text{wl}_{g^*}$-continuous.

**Proof:** Let $f$ be a continuous function and let $V$ be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is closed set in $(X, \tau, I)$. Since every closed set is $\text{wl}_{g^*}$-closed. Hence $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_{g^*}$-continuous.

**Example 2.6:** In example 2.4, let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_{g^*}$-continuous but not continuous.

**Theorem 2.7:** Ever $I_g$-continuous function is $\text{wl}_g$-continuous.

**Proof:** Let $f$ be an $I_g$-continuous function and let $V$ be a closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is $I_g$-closed set in $(X, \tau, I)$. Since every $I_g$-closed set is $\text{wl}_g$-closed. Hence $f^{-1}(V)$ is $\text{wl}_g$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_g$-continuous.

**Example 2.8:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\varphi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$, $\sigma = \{\varphi, \{d\}, \{c, d\}, Y\}$ and $I = \{\varphi, \{a\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be defined by $f(a) = b, f(b) = c, f(c) = a, f(d) = d$. Then the function $f$ is $\text{wl}_g$-continuous but not $I_g$-continuous.

**Theorem 2.9:** Ever $\tilde{g}$-continuous function is $\text{wl}_g$-continuous.

**Proof:** Let $f$ be an $\tilde{g}$-continuous function and let $V$ be a closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is $\tilde{g}$-closed set in $(X, \tau, I)$. Since every $\tilde{g}$-closed set is $\text{wl}_g$-closed set. Hence $f^{-1}(V)$ is $\text{wl}_g$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_g$-continuous.

**Example 2.10:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\varphi, \{b\}, \{a, b, c\}, X\}$, $\sigma = \{\varphi, \{c\}, \{a, c\}, Y\}$ and $I = \{\varphi, \{c\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_g$-continuous but not $\tilde{g}$-continuous.

**Theorem 2.11:** Ever $g$-continuous function is $\text{wl}_g$-continuous.

**Proof:** Let $f$ be a $g$-continuous function and let $V$ be a closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is $g$-closed set in $(X, \tau, I)$. Since every $g$-closed set is $\text{wl}_g$-closed set. Hence $f^{-1}(V)$ is $\text{wl}_g$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_g$-continuous.

**Example 2.12:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\varphi, \{b\}, \{c\}, \{b, c\}, X\}$, $\sigma = \{\varphi, \{c\}, X\}$ and $I = \{\varphi, \{b\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_g$-continuous but not $g$-continuous.

**Theorem 2.13:** Ever $I_{g^*}$-continuous function is $\text{wl}_{g^*}$-continuous.

**Proof:** Let $f$ be an $I_{g^*}$-continuous function and let $V$ be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is $I_{g^*}$-closed set in $(X, \tau, I)$. Since every $I_{g^*}$-closed set is $\text{wl}_{g^*}$-closed, hence $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_{g^*}$-continuous.

**Example 2.14:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\varphi, \{b\}, \{c\}, \{b, c\}, X\}$, $\sigma = \{\varphi, \{c\}, Y\}$ and $I = \{\varphi, \{d\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_{g^*}$-continuous but not $I_{g^*}$-continuous.

**Theorem 2.15:** Ever $g$-continuous function is $\text{wl}_{g^*}$-continuous.

**Proof:** Let $f$ be a $g$-continuous function and let $V$ be a closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is $g$-closed set in $(X, \tau, I)$. Since every $g$-closed set is $\text{wl}_{g^*}$-closed set. Hence $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_{g^*}$-continuous.

**Example 2.16:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\varphi, \{a\}, \{a, b, c\}, X\}$, $\sigma = \{\varphi, \{d\}, \{c, d\}, Y\}$ and $I = \{\varphi, \{a\}\}$. Let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_{g^*}$-continuous but not $g$-continuous.

**Theorem 2.17:** Ever $I_g$-continuous function is $\text{wl}_{g^*}$-continuous.

**Proof:** Let $f$ be an $I_g$-continuous function and let $V$ be a closed set in $(Y, \sigma)$, then $f^{-1}(V)$ is $I_g$-closed set in $(X, \tau, I)$. Since every $I_g$-closed set is $\text{wl}_{g^*}$-closed set. Hence $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_{g^*}$-continuous.

**Example 2.18:** In example 2.16, let the function $f : (X, \tau, I) \to (Y, \sigma)$ be the identity function. Then the function $f$ is $\text{wl}_{g^*}$-continuous but not $I_g$-continuous.

**Theorem 2.19:** Ever $I_g$-continuous function is $\text{wl}_{g^*}$-continuous.

**Proof:** Let $f$ be an $I_g$-continuous function and let $V$ be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is $I_g$-closed set in $(X, \tau, I)$. Since every $I_g$-closed set is $\text{wl}_{g^*}$-closed set. Hence $f^{-1}(V)$ is $\text{wl}_{g^*}$-closed set in $(X, \tau, I)$. Therefore $f$ is $\text{wl}_{g^*}$-continuous.

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Since every $I_\phi$-closed set is $w_{I_\phi}$-closed set. Hence $f^{-1}(V)$ is $w_{I_\phi}$-closed set in $(X, \tau, I)$. Therefore $f$ is $w_{I_\phi}$-continuous.

**Example 2.20:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{a,b,c\}, X\}$, $\sigma = \{\phi, \{a\}, \{a,c,d\}, Y\}$ and $I = \{\phi, \{d\}\}$. Let the function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ be the identity function. Then the function $f$ is $w_{I_\phi}$-continuous but not $I_\phi$-continuous.

**Theorem 2.21:** Ever $w_{I_\phi}$-continuous function is $w_{I_\phi}$-continuous.

**Proof:** Let $f$ be a $w_{I_\phi}$-continuous function and let $V$ be a closed set in $(Y, \sigma)$. Then $f^{-1}(V)$ is $w_{I_\phi}$-closed set in $(X, \tau, I)$. Since every $w_{I_\phi}$-closed set is $w_{I_\phi}$-closed. Hence $f^{-1}(V)$ is $w_{I_\phi}$-closed set in $(X, \tau, I)$. Therefore $f$ is $w_{I_\phi}$-continuous.

**Example 2.22:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a,b,c\}, X\}$, $\sigma = \{\phi, \{a\}, Y\}$ and $I = \{\phi, \{b\}\}$. Let the function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ be the identity function. Then the function $f$ is $w_{I_\phi}$-continuous but not $w_{I_\phi}$-continuous.

**Theorem 2.23:** A map $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is $w_{I_\phi}$-continuous iff the inverse image of every closed set in $(Y, \sigma)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$.

**Proof: Necessary:** Let $V$ be a closed set in $(Y, \sigma)$. Since $f$ is $w_{I_\phi}$-continuous, $f^{-1}(V)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$. By our assumption $f^{-1}(V) = X - f^{-1}(V)$, hence $f^{-1}(V)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$.

**Sufficiency:** Assume that the inverse image of every closed set in $(Y, \sigma)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$. Let $V$ be a closed set in $(Y, \sigma)$. By our assumption $f^{-1}(V) = X - f^{-1}(V)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$, which implies that $f^{-1}(V)$ is $w_{I_\phi}$-closed in $(X, \tau, I)$. Hence $f$ is $w_{I_\phi}$-continuous.

**Remark 2.24:**
(i) The union of any two $w_{I_\phi}$-continuous function is $w_{I_\phi}$-continuous.
(ii) The intersection of any two $w_{I_\phi}$-continuous function is need be not $w_{I_\phi}$-continuous.

**Theorem 2.25:** Let $f : (X, \tau, I_1) \rightarrow (Y, \sigma, I_2)$ and $g : (Y, \sigma, I_2) \rightarrow (Z, \eta, I_3)$ be any two functions. Then the following hold.
(i) $g \circ f$ is $w_{I_\phi}$-continuous if $f$ is $w_{I_\phi}$-continuous and $g$ is continuous.
(ii) $g \circ f$ is $w_{I_\phi}$-continuous if $f$ is $w_{I_\phi}$-irresolute and $g$ is continuous.
(iii) $g \circ f$ is $w_{I_\phi}$-irresolute if $f$ is $w_{I_\phi}$-irresolute and $g$ is irresolute.

**Proof:**
(i) Let $V$ be a closed set in $Z$. Since $g$ is continuous, $g^{-1}(V)$ is closed in $Y$. $w_{I_\phi}$-continuous of $f$ implies, $f^{-1}(g^{-1}(V))$ is $w_{I_\phi}$-closed in $X$ and hence $g \circ f$ is $w_{I_\phi}$-continuous.
(ii) Let $V$ be a closed set in $Z$. Since $g$ is $w_{I_\phi}$-irresolute, $g^{-1}(V)$ is $w_{I_\phi}$-closed in $Y$. Since $f$ is $w_{I_\phi}$-irresolute, $f^{-1}(g^{-1}(V))$ is $w_{I_\phi}$-closed in $X$. Hence $g \circ f$ is $w_{I_\phi}$-continuous.
(iii) Let $V$ be a $w_{I_\phi}$-closed set in $Z$. Since $g$ is $w_{I_\phi}$-irresolute, $g^{-1}(V)$ is $w_{I_\phi}$-closed in $Y$. Since $f$ is $w_{I_\phi}$-irresolute, $f^{-1}(g^{-1}(V))$ is $w_{I_\phi}$-closed in $X$. Hence $g \circ f$ is $w_{I_\phi}$-irresolute.

**Theorem 2.26:** Let $X = A \cup B$ be a topological space with topology $\tau$ and $Y$ be a topological space with topology $\sigma$. Let $f : (A, \tau/A) \rightarrow (Y, \sigma)$ and $g : (B, \tau/B) \rightarrow (Y, \sigma)$ be $w_{I_\phi}$-continuous maps such that $f(x) = g(x)$ for every $x \in A \cup B$. Suppose that $A$ and $B$ are $w_{I_\phi}$-closed sets in $X$. Then the combination $a : (X, \tau, I) \rightarrow (Y, \sigma)$ is $w_{I_\phi}$-continuous.

**Proof:** Let $F$ be any closed set in $Y$. Clearly $a^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. But $C$ is $w_{I_\phi}$-closed in $A$ and $D$ is be $w_{I_\phi}$-closed in $B$ and so $C$ is $w_{I_\phi}$-closed in $X$. Since we have proved that if $B \subseteq A \subseteq X$, $B$ is $w_{I_\phi}$-closed in $A$ and $B$ is $w_{I_\phi}$-closed in $X$, then $B$ is $w_{I_\phi}$-closed in $X$. Also $C \cup D$ is $w_{I_\phi}$-closed in $X$. Therefore $a^{-1}(F)$ is $w_{I_\phi}$-closed in $X$. Hence $a$ is $w_{I_\phi}$-continuous.
continuous maps such that \( f(x) = g(x) \) for every \( x \in A \cap B \).

Suppose that \( A \) and \( B \) are \( wIg \)-closed sets in \( X \). Then the combination \( \alpha : (X, \tau, I) \to (Y, \sigma) \) is \( wIg \)-continuous.

**Proof:** Let \( F \) be any closed set in \( Y \). Clearly \( \alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D \) where \( C = f^{-1}(F) \) and \( D = g^{-1}(F) \). But \( C \) is \( wIg \)-closed in \( A \) and \( A \) is be \( wIg \)-closed in \( X \) and so \( C \) is \( wIg \)-closed in \( X \). Since we have proved that if \( B \subseteq A \subseteq X \), \( B \) is \( wIg \)-closed in \( A \) and \( A \) is \( wIg \)-closed in \( X \), then \( B \) is \( wIg \)-closed in \( X \). Also \( C \cup D \) is \( wIg \)-closed in \( X \). Therefore \( \alpha^{-1}(F) \) is \( wIg \)-closed in \( X \). Hence \( \alpha \) is \( wIg \)-continuous.

**References**

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