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Baljit Singh
Department of Mathematics,
Guru Jambheshwar University
of Science and Technology
Hisar, Haryana, India

Correspondence
Baljit Singh
Department of Mathematics,
Guru Jambheshwar University
of Science and Technology
Hisar, Haryana, India

Study of wave propagation with two phase lag theory

Baljit Singh

Abstract
In the present paper, we have introduced the two phase lag theory to study the dynamical interactions in a thermoelastic medium. Normal mode analysis technique is employed onto the non-dimensional field equations to obtain the analytical solution. The numerical estimates of the field variables displacement, stress and temperature are computed for magnesium crystal like material and presented graphically.

Keywords: Two phase lag, thermoelasticity, normal mode analysis, inclined load

1. Introduction
Abbreviation

σ_{ij}	Components of force stress tensor
θ	$T - T_0$
T	Absolute temperature
T_0	Temperature of the medium in its natural state assumed to be $\left\lvert \frac{\theta}{T_0} \right\rvert << 1$
λ, μ	Lame's constants
β_1	$(3\lambda + 2\mu)\alpha_t$
α_t	Coefficient of linear thermal expansion
u_i	Components of the displacement vector
ρ	Density of the medium
e	Cubical dilatation
C_E	Specific heat at constant strain
k^*	Thermal conductivity
τ_T	Phase lag for the temperature gradient
τ_{ij}	Phase lag for heat flux

The classical theory of thermoelasticity suffers from deficiency of admitting thermal signals propagating with infinite speed. Numerous alternative theories of heat conduction have come forth to overcome this deficiency. Generalized theories proposed by Lord and Shulman (1967) [6] and Green and Lindsay (1972) [2] are two well known theories of thermoelasticity to overcome this deficiency. After that, providing sufficient basic modifications in governing equations, Green and Naghdi (1991, 1992, 1993) [3, 4, 5] produced an alternative theory which was further divided into three different parts, referred to as GN theory of type I, II, III. The conventional Fourier's equation of heat conduction $\vec{q}(\vec{r}, t) = -k \nabla \vec{T}(\vec{r}, t)$ can be used in several particular problems, although this turns out to predict an infinite speed of thermal signal, which is physically unrealistic. (Tzou, 1995) [8] developed dual phase lag theory to overcome this deficiency. The classical Fourier's law $\vec{q} = -k \nabla \vec{T}$ has been replaced by $\vec{q}(P, t + \tau_q) = -k \nabla \vec{T}(P, t + \tau_T)$, where the temperature gradient $\nabla \vec{T}$ at a space variable *P* at time $t+\tau_T$ corresponds to heat flux vector $\vec{q}$ at the same point at time $t+\tau_q$. The delay time τ_T is supposed to be caused by the microstructural interactions and is called the phase lag of temperature gradient. The other delay time τ_q is interpreted as the relaxation time due to the fast transient effects of the thermal inertia and is called the phase lag of the heat flux. If $\tau_q = \tau_T = 0$, then the Fourier's law in two phase lag model is identical with the classical Fourier's law. The stability of dual phase lag heat conduction was discussed by (Quintanilla and Racke, 2006). (Ezzat *et al*. 2012) [1] estimated the effects of two-temperature discrepancy and fractional parameter on the wave propagation in the context of dual-phase-lag magneto-thermoelasticity.

The present investigation is concerned with the determination of displacement components and force stresses generalized thermoelastic medium with dual phase lag effects subjected to inclined mechanical load. Normal mode analysis is adopted to find out the exact expressions of the variables considered.

2. Governing equations

The constitutive relations and field equations in the absence of body force for an isotropic, homogeneous, thermoelastic solid can be expressed as follows:

(i) Constitutive relations

$$\sigma_{ij} = \lambda u_{r,r}\delta_{ij} + \mu\left(u_{j,i} + u_{i,j}\right) - \beta_1\theta\delta_{ij}, \tag{1}$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2}$$

(ii) Stress equation of motion

$$\sigma_{ji,j} = \rho\ddot{u}_i, \tag{3}$$

(iii) Equation of heat conduction

$$k^*\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2\theta = \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E\dot{\theta}+\beta_1 T_0\dot{e}\right). \tag{4}$$

3. Problem formulation

Let us consider a homogeneous, isotropic, generalized thermoelastic medium. The rectangular Cartesian co-ordinates are introduced having origin on the surface ($z = 0$) and z-axis pointing vertically downwards into the medium. All the quantities related to the medium considered will be functions of the time variable t and the coordinates x and z. For a two dimensional problem in the x-z plane, we can write the displacement vector as

$$u = u_x = u(x,z,t),\quad v = u_y = 0, \text{ and } w = u_z = w(x,z,t). \tag{5}$$

Substitution of Eqs. (2) and (5) into Eqs. (1)

$$\sigma_{xx} = (2\mu+\lambda)\frac{\partial u}{\partial x} + \lambda\frac{\partial w}{\partial z} - \beta_1\theta, \tag{6}$$

$$\sigma_{zz} = (2\mu+\lambda)\frac{\partial w}{\partial z} + \lambda\frac{\partial u}{\partial x} - \beta_1\theta, \tag{7}$$

$$\sigma_{xz} = \sigma_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{8}$$

With the aid of expressions (1), (2) and (5), the equations of motion (3), take the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \mu\nabla^2 u + (\mu+\lambda)\frac{\partial^2 u}{\partial x^2} + (\mu+\lambda)\frac{\partial^2 w}{\partial z\partial x} - \beta_1\frac{\partial\theta}{\partial x}, \tag{9}$$

$$\rho\frac{\partial^2 w}{\partial t^2} = \mu\nabla^2 w + (\mu+\lambda)\frac{\partial^2 w}{\partial z^2} + (\mu+\lambda)\frac{\partial^2 u}{\partial z\partial x} - \beta_1\frac{\partial\theta}{\partial z}, \tag{10}$$

The governing equations can be put into more convenient forms by introducing the following non-dimensional variables:

$$(x', z') = \frac{w^*}{c_1}(x, z), \quad (u', w') = \frac{\rho w^* c_1}{\beta_1 T_0}(u, w),$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\beta_1 T_0}, \quad (t', \tau'_T, \tau'_q) = w^*(t, \tau_T, \tau_q),$$

$$\theta' = \frac{\theta}{T_0}, \tag{11}$$

where

$$w^* = \frac{\rho C_E c_1^2}{k^*}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}.$$

Using Helmholtz decomposition, the displacement components can be written as

$$u = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x}, \quad \psi = (-\vec{U})_y, \tag{12}$$

where $q(x, z, t)$ and $\psi(x, z, t)$ are scalar potential functions and $\vec{U}(x, z, t)$ is the vector potential function.
Now, in terms of the dimensionless quantities given in (11), Eqs. (9)-(10), (4) and (6)-(8) with the aid of expressions (12) along with some simplifications, assume the forms (after dropping the primes)

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) q - \theta = 0, \tag{13}$$

$$\left(\nabla^2 - a_1 \frac{\partial^2}{\partial t^2}\right) \psi = 0, \tag{14}$$

$$k^* \left(1 + \tau_T \frac{\partial}{\partial t}\right) \nabla^2 \theta = \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2} \frac{\partial^2}{\partial t^2}\right) \frac{\partial}{\partial t} \left(\theta + \delta_0 \nabla^2 q\right), \tag{15}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} + a_1 \frac{\partial w}{\partial z} - \theta, \tag{16}$$

$$\sigma_{zz} = \frac{\partial w}{\partial z} + a_4 \frac{\partial u}{\partial x} - \theta, \tag{17}$$

$$\sigma_{xz} = \sigma_{zx} = a_2 \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \tag{18}$$

where

$$a_1 = \frac{\rho c_1^2}{\mu}, \quad \delta_0 = \frac{\beta_1^2 T_0}{\rho}, \quad a_2 = \frac{\lambda}{\rho c_1^2}, \quad a_3 = \frac{1}{a_1}.$$

4. Solution of the problem

Solution of the physical quantities can be decomposed in terms of normal modes in the following form

$$\left(u, w, q, \psi, \theta, \sigma_{ij}\right)(x, z, t) = \left(u^*, w^*, q^*, \psi^*, \theta^*, \sigma_{ij}^*\right)(z)\, e^{(\omega t + \iota m x)}, \tag{19}$$

where $u^*(z), w^*(z), q^*(z), \psi^*(z), \phi_2^*(z), \theta^*(z), \sigma_{ij}^*(z)$ are the amplitudes of the functions, ω is the angular frequency, ι is the imaginary unit and *m* is the wave number in *x* direction.
Using (19) in the Eqs. (13)-(15), we obtain the following differential equations

$$\left(D^2 - A_1\right) \psi^*(z) = 0, \tag{20}$$

$$\left(D^4 - A_2 D^2 + A_3\right)\left\{q^*(z), \theta^*(z)\right\} = 0, \tag{21}$$

where $D = \frac{d}{dz}$, $A_1 = m^2 + a_1 w^2$, $A_2 = b_2 + b_3 + b_4$, $A_3 = b_2 b_3 + b_4 m^2$,

$b_4 = b_1$, $b_3 = m^2\delta_0$, $b_2 = m^2 + b_1$, $b_1 = \frac{\left(1+\tau_q\omega+\frac{\tau_q^2\omega^2}{2}\right)\omega}{k^*(1+\tau_T\omega)}$,

Since the intent is that the solutions vanish at infinity so as to satisfy the regularity condition at infinity (which are assumed to be bounded as $z \to \infty$), we can express $\psi^*(x,z,t)$, $q^*(x,z,t)$, $\theta^*(x,z,t)$ in the following forms:

$$\psi(x,z,t) = \left(H_{1i}R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (22)$$

$$q(x,z,t) = \left(\sum_{i=2}^{3} R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (23)$$

$$\theta(x,z,t) = \left(\sum_{i=2}^{3} H_{1i}R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (24)$$

where λ_1^2 and λ_2^2, λ_3^2 are roots with positive real parts of the characteristic equations (20) and (21) respectively, $R_i(m,\omega)$, $(i = 1,2,3)$ are parameters, depending upon *m* and ω, and

$H_{11} = -\left(\frac{a_2}{\lambda_i^2 - b_2}\right)$, $H_{1i} = (\lambda_i^2 - b_1)(i = 2,3)$.

Application on normal mode analysis to the expressions for stress components (16), (18) and displacement components (12), in combination with the relations (22)-(24), yields

$$u(x,z,t) = \left(\sum_{i=1}^{3} H_{2i}R_i e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (24)$$

$$w(x,z,t) = \left(\sum_{i=1}^{3} H_{3i}R_i e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (26)$$

$$\sigma_{zx}(x,z,t) = \left(\sum_{i=1}^{3} H_{4i}R_i e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (27)$$

$$\sigma_{zz}(x,z,t) = \left(\sum_{i=1}^{3} H_{5i}R_i e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \quad (28)$$

where $H_{21} = -\lambda_1 H_{11}$, $H_{2i} = \iota m\ (i = 2,3)$, $H_{31} = -\iota m H_{11}$

$H_{3i} = -\lambda_i\ (i = 2,3)$, $H_{41} = (a_{10}\lambda_1^2 + a_{11}m^2)H_{11} - a_{12}$

$H_{4i} = -(a_{10} + a_{11})m\iota\lambda_i\ (i = 2,3)$, $H_{51} = (1 - a_9)\iota m\lambda_1 H_{11}$

$H_{5i} = \lambda_i^2 - a_9 m^2 - H_{1i}\ (i = 2,3)$.

5. Application: Inclined load acting on the surface

The plane boundary is subjected to an inclined mechanical load P_0 and its inclination with *z*-axis is θ. Then we have $P_1 = P_0\sin\theta$, $P_2 = P_0\cos\theta$, where P_1 is the intensity of tangential line load acting at the origin in the positive *x*-direction and P_2 is the intensity of normal line load acting in the positive *z*-direction. Hence the boundary conditions are

$$\sigma_{zz}(x,0,t) = -P_2\psi_1(x,t), \quad (29)$$

$$\sigma_{zx}(x,0,t) = -P_1\psi_1(x,t), \quad (30)$$

$$\theta(x,0,t) = 0. \quad (31)$$

where $\psi_1(x,t) = \delta(x)H(t)$, $\delta(x)$ is Dirac delta function and $H(t)$ is Heaviside unit step function.

$$\vec{\sigma}_{zz}(m,0,\omega) = \frac{-P_0\cos\theta}{\omega} \quad (31)$$

$$\bar{\sigma}_{zx}(m,0,\omega) = \frac{-P_0 \sin\theta}{\omega}, \quad (32)$$

$$\bar{\theta}(m,0,\omega) = 0, \quad (33)$$

We arrive at a non-homogeneous system of linear equations which can be written in the matrix form as

$$\begin{bmatrix} 0 & H_{12} & H_{13} \\ H_{41} & H_{42} & H_{43} \\ H_{51} & H_{52} & H_{53} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ 0 \end{bmatrix} \quad (32)$$

Solution of system (32) provides us the values of $R_i\,(i = 1,2,3)$ as:

$$R_1 = \frac{\Delta_1}{\Delta}, \quad R_2 = \frac{\Delta_2}{\Delta}, \quad R_3 = \frac{\Delta_3}{\Delta}, \quad (33)$$

where

$$\Delta = H_{51}(H_{12}H_{43} - H_{13}H_{42}) - H_{41}(H_{12}H_{53} - H_{13}H_{52}),$$

$$\Delta_1 = M_1(H_{42}H_{53} - H_{43}H_{52}) - M_2(H_{12}H_{53} - H_{13}H_{52}),$$

$$\Delta_2 = -M_1(H_{41}H_{53} - H_{43}H_{51}) - M_2(H_{13}H_{51}),$$

$$\Delta_3 = M_1(H_{41}H_{52} - H_{42}H_{51}) + M_2(H_{12}H_{51}), M_1 = \frac{-P_0\cos\theta}{\omega}, \quad M_2 = \frac{-P_0\sin\theta}{\omega}$$

Substitution of (33) into expressions (24) and (28) leads to the expression of field variables as:

$$\theta(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3} \Delta_i H_{1i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)}, \quad (34)$$

$$u(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3} \Delta_i H_{2i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \quad (35)$$

$$w(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3} \Delta_i H_{3i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \quad (36)$$

$$\sigma_{zx}(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3} \Delta_i H_{4i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \quad (37)$$

$$\sigma_{zz}(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3} \Delta_i H_{5i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \quad (38)$$

6. Numerical results and discussions

The dynamical interactions between thermal and mechanical fields in solids have many applications in aeronautics, nuclear reactors and high energy particle accelerators. To understand the interaction phenomena, we have evaluated the numerical results of non-dimensional displacement component *w*, normal stress σ_{zz}, and temperature θ and displayed graphically. For numerical computation, we take the following values of relevant parameters for magnesium crystal like material:

$$\rho = 1.74\times10^3\,kg\,m^{-3},\quad k = 1.0\times10^{10}\,kg\,m^{-1}\,s^{-2},$$

$$k^* = 2.510\,W m^{-1}k^{-1},\quad C_E = 9.623\times10^2\,J\,kg^{-1}k^{-1},$$

$$\alpha_t = 2.36\times10^{-5}\,K^{-1},\quad T_0 = 293\,K,$$

$$\tau_q = 0.2\,s,\quad \tau_T = 0.15\,s. \quad (39)$$

Utilizing the above values of parameters, values of the non-dimensional field variables have been evaluated and the results are displayed in the form of the graphs at different positions of *z* at *t* = 0.01 and *x* = 1.0. (Figs. 1-2) depicts the effect of inclination of load on the distribution of the field variables by considering three different values of angle as $\theta = 0°$ (solid line), $\theta = 45°$ (long-dashed line) and $\theta = 90°$ (small-dashed line).

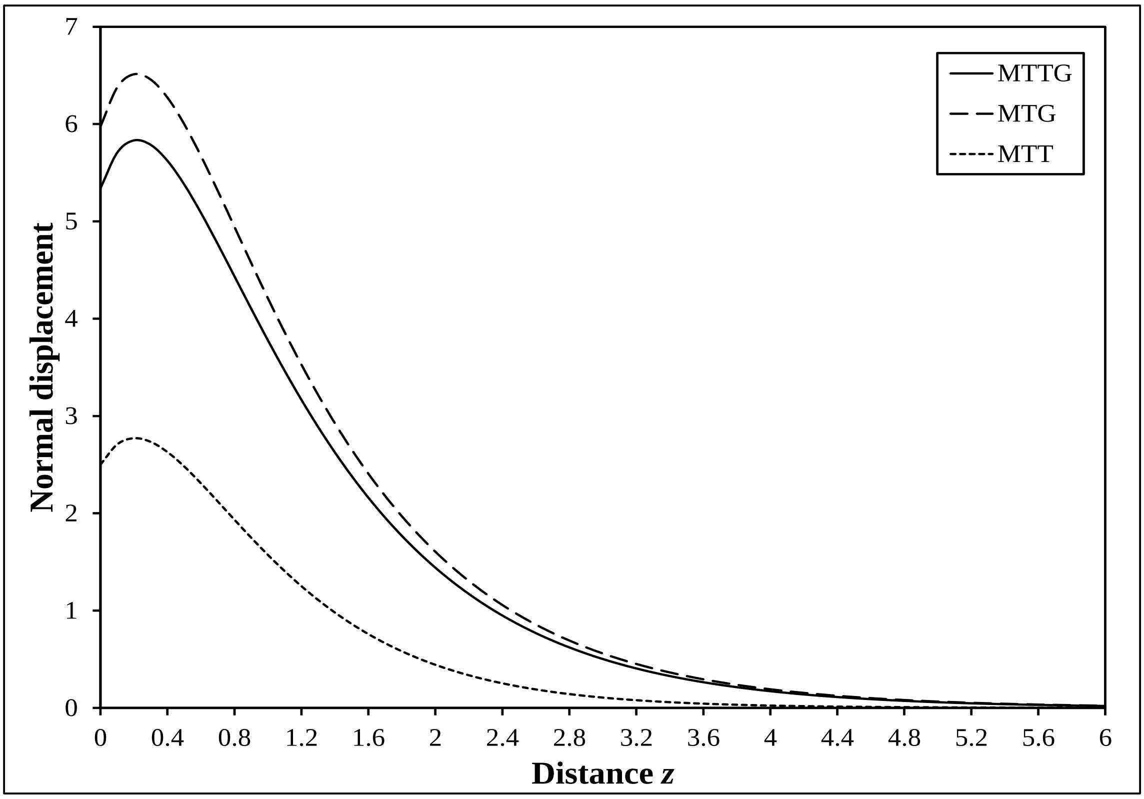

Fig 1

Figure 1 represents the spatial variations of displacement component *w* with distance *z* for different values of angle taking the distance axis as $0 \le z \le 5.8$. In the beginning, for angle $\theta = 45°$ and $90°$, the vibration amplitude quickly rises to its maximum value, which is called the peak deflection. Then the amplitude damps gradually to near zero. At angle $\theta = 0°$, there is a continuous decrease in the values of displacement field. Figure 2 is plotted to observe the variations in the thermodynamical temperature θ with distance *z*. The thermodynamical temperature is having the similar pattern in all the three cases. The numerical values of temperature field at angle $\theta = 0°$ are small in comparison with those at angles $\theta = 45°$ and 90°. The impact dies out as we move away from the boundary.

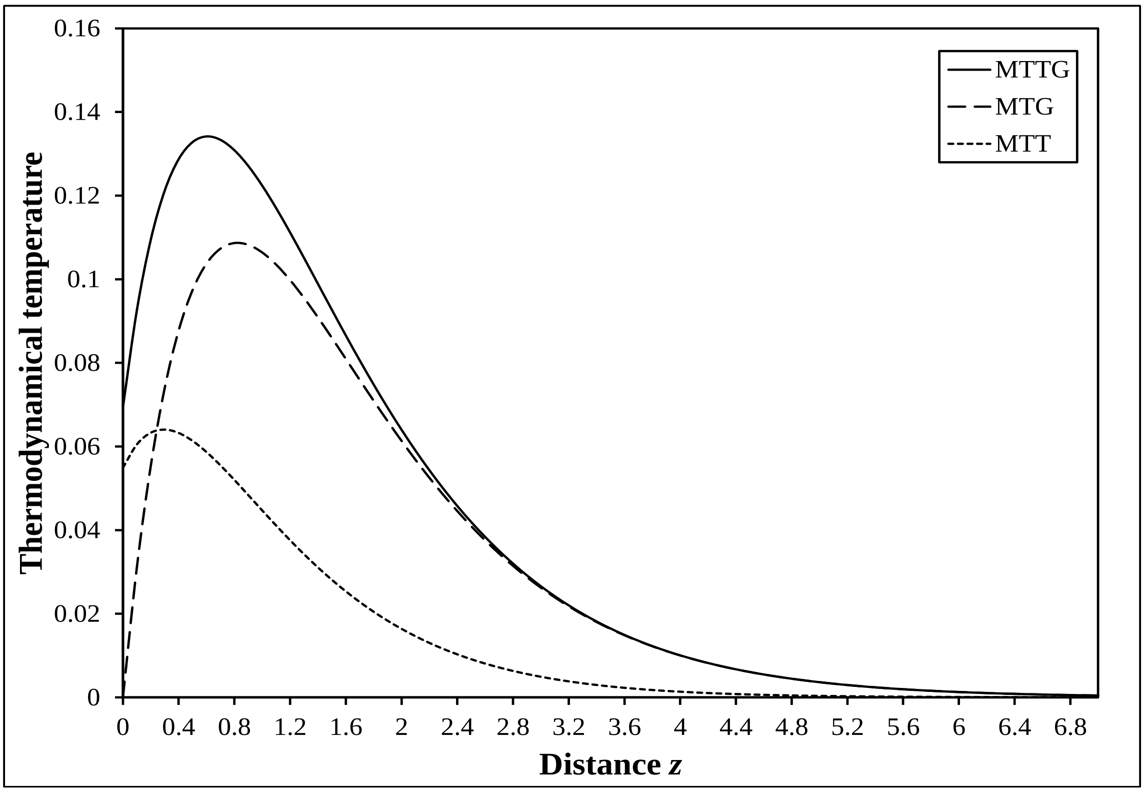

Fig 2

7. Concluding remarks

The work presented in this article provides a mathematical model to obtain the behaviour of normal displacement and temperatures in a homogeneous, isotropic, thermoelastic medium with two dual phase lag theory by using normal mode analysis:

i. In all the figures, it is clear that all the fields are restricted to a limited region, which is in accordance with the notion of generalized thermoelasticity theory and support the physical facts.
ii. The effect of angle of inclination of load on all the studied fields is very much significant.

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