Characterizations of mildly nano gb-normal spaces

Dhanis Arul Mary A, Arockiarani I

Abstract
The purpose of this paper is to introduce a new class of nano-normal spaces in a nano topological space. We obtain the relationships of such normal spaces, and present some properties and establish various preservation theorems.

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1. Introduction
Normality is an important topological property which was initially taken by Vigilino [20] who defined semi normal spaces in the year 1971. Then Signal and Arya [19] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a semi-normal space and an almost normal space. In recent years, many others have studied several forms of normality [9, 10, 15, 17]. On the other hand, the notions of p-normal spaces, s-normal spaces were introduced by Paul and Bhattacharya [18]. Levine [11] initiated the investigation of g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by number of topologists [4-7]. Ahmad Al-Omari and Mohd. Salmi Md. Noorani [1] introduced the concepts of gb-closed sets. In 2013, the concept of nano topology was introduced by Lellis Thivagar [12], which was defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it.

The aim of this paper is to introduce a new class of different normal spaces, namely nano gb-normal spaces, mildly nano gb-normal spaces and almost nano gb-normal spaces in nano topological spaces. The relations among nano b-normal spaces, nano p-normal spaces, nano s-normal spaces and also properties of nano gb-normal spaces are investigated. Moreover, we study new forms of nano gb-closed functions and obtain properties of these new forms of nano gb-closed functions and preservation theorems.

2. Preliminaries
Throughout this paper \((U, \tau_g(X))\) and \((V, \tau_g(Y))\) mean nano topological spaces on which no separation axioms are assumed unless explicitly stated. The nano closure of \(A\) and the nano interior of \(A\) are denoted by \(Ncl(A)\) and \(Nint(A)\) respectively.

**Definition 2.1** [12]: Let \(U\) be a non-empty finite set of objects called the universe and \(R\) be an equivalence relation on \(U\) named as the indiscernibility relation. Then \(U\) is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair \((U, R)\) is said to be the approximation space. Let \(X \subseteq U\).

1. The lower approximation of \(X\) with respect to \(R\) is the set of all objects, which can be for certainly classified as \(X\) with respect to \(R\) and it is denoted by \(L_R(X)\). That is

\[L_R(X) = \bigcup_{x \in X} \{R(x) : R(x) \subseteq X\},\]

where \(R(x)\) denotes the equivalence class determined by \(x \in U\).
2. The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ with respect to $R$ and it is denoted by $U_R(X)$. That is
\[ U_R(X) = \bigcup_{x \in X} \{ R(x) : R(x) \cap X \neq \emptyset \} \]

3. The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not-$X$ with respect to $R$ and it is denoted by $B_R(X)$. That is
\[ B_R(X) = U_R(X) - L_R(X). \]

**Definition 2.2** \[^{[12]}\]: Let $U$ be non-empty, finite universe of objects and $R$ be an equivalence relation on $U$. Let $X \subseteq U$. Let $\tau_R(X) = \{ U, \emptyset, L_R(X), U_R(X), B_R(X) \}$. Then $\tau_R(X)$ is a topology on $U$, called as the nano topology with respect to $X$. Elements of the nano topology are known as the nano-open sets in $U$ and $(U, \tau_R(X))$ is called the nano topological space. $[\tau_R(X)]^c$ is called as the dual nano topology of $\tau_R(X)$. Elements of $[\tau_R(X)]^c$ are called as nano closed sets.

**Definition 2.3** \[^{[12]}\]: Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. Then $A$ is said to be
(i) nano semi-open if $A \subseteq Ncl(Nint(A))$
(ii) nano pre-open if $A \subseteq Nint(Ncl(A))$
(iii) nano $\alpha$-open if $A \subseteq Nint(Ncl(Nint(A)))$
(iv) nano semi-pre-open if $A \subseteq Nbcl(Ncl(A))$
(v) nano $\beta$-open if $A \subseteq Ncl(Ncl(Ncl(A)))$.

**NSO(U, X), NPO(U, X), Ncl(U, X), NSPO(U, X) and NBO(U, X) respectively denote the families of all nano open, nano semi-open, nano pre-open, nano $\alpha$-open, nano semi-pre-open and nano $\beta$-open subsets of $U$.**

Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. $A$ is said to be
- nano semi closed, nano pre closed, nano $\alpha$-closed, nano semi pre closed and nano $\beta$-closed if its complement is respectively nano semi closed, nano pre closed, nano $\alpha$-closed, nano semi pre closed and nano $\beta$-closed.

**Definition 2.4** \[^{[6]}\]: A subset $A$ of a nano topological space $(U, \tau_R(X))$ is called a nano generalized $b$-closed (briefly, nano gb-closed), if $Nbcl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is nano open in $U$.

**Definition 2.5** \[^{[13]}\]: A function $f : (U, \tau_R(X)) \to (V, \tau_K(Y))$ is called a nano continuous if the inverse image of every nano closed set in $(V, \tau_K(Y))$ is nano closed in $(U, \tau_R(X))$.

**Definition 2.6** \[^{[7]}\]: A function $f : (U, \tau_R(X)) \to (V, \tau_K(Y))$ is called a nano gb-irresolute if the inverse image of every nano closed set in $(V, \tau_K(Y))$ is nano gb-closed in $(U, \tau_R(X))$.

**Definition 2.7**: A space $X$ is said to be $p$-normal \[^{[17]}\] (resp. $s$-normal \[^{[14]}\]) if for any pair of disjoint closed sets $A$ and $B$, there exist disjoint preopen (resp. semi open) sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 2.8**: A nano topological space $(U, \tau_R(X))$ is said to nano $b$-normal if for any pair of disjoint nano closed sets $A$ and $B$, there exist disjoint nano b-open sets $M$ and $N$ such that $A \subseteq M$ and $B \subseteq N$.

3. **Nano gb-normal spaces**

**Definition 3.1**: A nano topological space $(U, \tau_R(X))$ is said to be nano gb-normal if for any pair of disjoint nano closed sets $A$ and $B$, there exist disjoint nano gb-open sets $M$ and $N$ such that $A \subseteq M$ and $B \subseteq N$.

**Remark 3.2**: The following diagram holds for a nano topological space $(U, \tau_R(X))$.

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1 --3
2
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None of these implications is reversible as shown by the following examples.

**Example 3.3**: Let $U = \{a, b, c, d\}$ with $U/R = \{\{d\}, \{a, b, c\}, \{b, c, d\}\}$ and $X = \{b, d\}$. The nano topology is defined as $\tau_R(X) = \{ U, \emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. Then $\tau_R(X)$ is nano $b$-normal but not nano $s$-normal.

**Example 3.4**: Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{a, c\}, \{a, b, d\}\}$ and $X = \{a, b\}$. The nano topology is defined as $\tau_R(X) = \{ U, \emptyset, \{a, b, d\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\tau_R(X)$ is nano $b$-normal but not nano p-normal.

**Example 3.5**: Let $U = \{a, b, c, d\}$ with $U/R = \{\{d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $X = \{b, d\}$. The nano topology is defined as $\tau_R(X) = \{ U, \emptyset, \{a, b, d\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\tau_R(X)$ is nano p-normal but not nano normal.

**Theorem 3.6**: For a nano topological space $(U, \tau_R(X))$ the following are equivalent:
1) $U$ is nano gb-normal,
2) For every pair of nano open sets $M$ and $N$ whose union is $U$, there exist nano gb-closed sets $A$ and $B$ such that $A \subseteq M$ and $B \subseteq N$ and $A \cup B = U$,
3) For every nano closed set $H$ and every nano open set $K$ containing $H$, there exists a nano gb-open set $M$ such that $H \subseteq M \subseteq \text{Ngb-cl}(M) \subseteq K$.
Proof: (1) \(\Rightarrow\) (2): Let M and N be a pair of nano open sets in a nano gb-normal space U such that U = M \(\cup\) N. Then, U \(\setminus\) M, U \(\setminus\) N are disjoint nano closed sets. Since U is nano gb-normal, there exist nano gb-open sets \(M_1\) and such \(N_1\) such that U \(\setminus\) M \(\subseteq\) \(M_1\) and U \(\setminus\) N \(\subseteq\) \(N_1\). Let A = U \(\setminus\) \(M_1\), \(B = U \setminus N_1\). Then A and B are nano gb-closed sets such that A \(\subseteq\) M and B \(\subseteq\) N and A \(\cup\) B = U.

(2) \(\Rightarrow\) (3) Let H be a nano closed set and K be a nano open set containing H. Then U \(\setminus\) H and K are nano open sets in a nano gb-normal space U such that U = M \(\cup\) N. Then M and N are disjoint nano gb-open sets such that H \(\subseteq\) M \(\subseteq\) \(M_1\) \(\cap\) (U \(\setminus\) P1) \(\cap\) (U \(\setminus\) P2) = \(\phi\). Let M = U \(\setminus\) P1 and N = U \(\setminus\) P2. Then M and N are disjoint nano gb-open sets such that \(\phi\) \(\subseteq\) H \(\subseteq\) M \(\subseteq\) \(M_1\) \(\subseteq\) \(\phi\).

Definition 3.11: A function \(f : (U, \tau_g(X)) \rightarrow (V, \tau_g(Y))\) is a nano \(\alpha\) - closed if for each nano closed set M in U, f(M) is nano \(\alpha\) - closed in V.

Theorem 3.12: If \(f : (U, \tau_g(X)) \rightarrow (V, \tau_g(Y))\) is an nano \(\alpha\) - closed continuous surjection and U is nano normal, then V is nano gb-normal.

Proof: Let K1 and K2 be disjoint nano closed sets in V. Then \(f^{-1}(K_1)\) and \(f^{-1}(K_2)\) are nano closed sets. Since U is nano gb-normal, there exist disjoint nano gb-open sets M and N such that \(f^{-1}(K_1)\) \(\subseteq\) M and \(f^{-1}(K_2)\) \(\subseteq\) N. By Theorem 3.6, there exist nano gb-open sets A and B such that \(K_1 \subseteq A\), \(K_2 \subseteq B\), \(f^{-1}(A)\) \(\subseteq\) M and \(f^{-1}(B)\) \(\subseteq\) N. Also, A and B are disjoint. Thus, V is nano gb-normal.
Theorem 4.2: \( f(M) \) is a nano topological space.

Definition 4.1: 4. Almost and mildly nano gb-normal Spaces

Theorem 4.8: For a nano topological space \( (U, \tau_g(X)) \) the following are equivalent:
1) \( U \) is mildly nano gb-normal,
2) For every pair of nano regular open sets \( M \) and \( N \), whose union is \( U \), there exist disjoint regular open sets \( A \) and \( B \) such that \( A \subset M \) and \( B \subset N \).
3) For every nano regular closed set \( A \) and every nano regular open set \( B \) containing \( A \), there exists a nano gb-open set \( N \) such that \( A \subset N \) and \( N \subset \text{Ngbc}(U) \).
4) For every pair of disjoint nano regular closed sets \( A \) and \( B \), there exists a nano gb-open set \( N \) such that \( A \subset N \) and \( N \subset \text{Ngbc}(U) \).

Proof: Similar to Theorem 3.6.

Corollary 4.5: If \( f : (U, \tau_g(X)) \to (V, \tau_g(Y)) \) is a completely nano continuous strongly nano gb-open and almost nano gb-irresolute surjection from an almost nano gb-normal space \( U \) onto a space \( V \), then \( V \) is almost nano gb-normal.

Definition 4.6: A nano topological space \( (U, \tau_g(X)) \) is said to be mildly nano gb-normal if for every pair of disjoint nano regular closed sets \( A \) and \( B \) of \( U \), there exist disjoint nano gb-open sets \( M \) and \( N \) such that \( A \subset M \) and \( B \subset N \).

Remark 4.7: The following diagram holds for a nano topological space \( (U, \tau_g(X)) \):

\[
\begin{array}{c}
\text{nano gb-normal} \quad \text{almost nano gb-normal} \quad \text{mildly nano gb-normal}
\end{array}
\]

None of the above implications are reversible.
Proof: Similar to Theorem 3.10.

Theorem 4.11: Let \( f : (U, \tau_g(X)) \rightarrow (V, \tau_f(Y)) \) be a nano continuous quasi nano gb-closed surjection and \( U \) is nano gb-normal, then \( V \) is nano gb-normal.

Proof: Let \( P_1 \) and \( P_2 \) be any disjoint nano closed sets of \( V \). Since \( f \) is nano continuous, \( f^{-1}(P_1) \) and \( f^{-1}(P_2) \) are disjoint nano closed sets of \( U \). Since \( U \) is nano gb-normal, there exist disjoint \( M_1, M_2 \in \text{NgbO}(U) \) such that \( f^{-1}(P_1) \subset M_i \) for \( i = 1, 2 \). Put \( N_i = V \setminus f(U \setminus M_i) \), then \( N_i \) is nano open in \( V \), \( P_i \subset N_i \) and \( f^{-1}(N_i) \subset M_i \) for \( i = 1, 2 \). Since \( M_1 \cap M_2 = \phi \) and \( f \) is surjective, we have \( N_1 \cap N_2 = \phi \). This shows that \( V \) is nano gb-normal.

Corollary 4.16: Let \( f : (U, \tau_g(X)) \rightarrow (V, \tau_f(Y)) \) be a nano continuous nano gb-closed surjection. If \( U \) is nano normal, then \( V \) is nano gb-normal.

5. References