The invariance of the Hankel transform under K-Binomial transform of a sequence

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Abstract
We give a new proof of the invariance of the Hankel transform under the binomial transform of a sequence. Our method of proof leads to three variations of the binomial transform; we call these the k-binomial transforms. We give a simple means of constructing these transforms via a triangle of numbers. We show how the exponential generating function of a sequence changes after our transforms are applied, and we use this to prove that several sequences in the On-Line Encyclopedia of Integer Sequences are related via our transforms.
In the process, we prove three conjectures in the OEIS. Addressing a question of Layman, we then show that the Hankel transform of a sequence is invariant under one of our transforms, and we show how the Hankel transform changes after the other two transforms are applied. Finally, we use these results to determine the Hankel transforms of several integer sequences.

Keywords: Hankel transform, K-Binomial transform.

1. Introduction
Given a sequence $A = \{a_0, a_1, \ldots\}$, define the binomial transform $B$ of a sequence $A$ to be the sequence $B(A) = \{b_n\}$, where $b_n$ is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.
$$

Define the Hankel matrix of order $n$ of $A$ to be the $(n + 1) \times (n + 1)$ upper left sub-matrix of

$$
\begin{bmatrix}
    a_0 & a_1 & a_2 & a_3 & \cdots \\
    a_1 & a_2 & a_3 & a_4 & \cdots \\
    a_2 & a_3 & a_4 & a_5 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

Let $h_n$ denote the determinant of the Hankel matrix of order $n$. Then define the Hankel transform $H$ of $A$ to be the sequence $H(A) = \{h_0, h_1, h_2, \ldots\}$. For example, the Hankel matrix of order 3 of the derangement numbers, $\{D_n\} = \{1, 0, 1, 2, 9, 44, 265, \ldots\}$, is

$$
\begin{bmatrix}
    1 & 0 & 1 & 2 \\
    0 & 1 & 2 & 9 \\
    1 & 2 & 9 & 44 \\
    2 & 9 & 44 & 265
\end{bmatrix}
$$

The determinant of this matrix is 144, which is $(0!)2 (1!)2 (2!)2 (3!)2$. 

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The interpretations of the transforms as well. We show how the exponential generating function of a sequence changes after applying our transforms, and we use these results to prove that several sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) are related by the transforms.

In the process, we prove three conjectures listed in the OEIS concerning the binomial mean transform. Then, giving an answer to Layman’s question, we show that the Hankel transform is invariant under the falling k-binomial transform. The Hankel transform is not invariant under the k-binomial transform and the rising k-binomial transform, but we give a formula showing how the Hankel transform changes under these two transforms.

These results, together with our proofs of relationships between sequences in the OEIS, determine the Hankel transforms of several sequences in the OEIS. (Unfortunately, this is some discrepancy in the literature in the definitions of the Hankel determinant and the binomial transform of an integer sequence.

The definitions of the Hankel determinant in Layman and in Ehrenborg are slightly different from each other, and ours is slightly different from both of these. As many sequences of interest begin indexing with 0, we define the Hankel determinant and the Hankel transform so that the first elements in the sequences A and H(A) are indexed by 0. Layman’s definition begins indexing A and H(A) by 1, whereas Ehrenborg’s definition results in indexing A beginning with 0 and H(A) beginning with 1. Our definition of the binomial transform is that used by Layman and the OEIS.

The K-Binomial Transforms

We now consider three variations of the binomial transform. All three transforms take two parameters: the input sequence A and a scalar k. The k-binomial transform W of a sequence A is the sequence W(A, k) = {wn}, where wn is given by

\[ w_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i. \]

The rising k-binomial transform R of a sequence A is the sequence R(A, k) = {rn}, where rn is given by

\[ r_n = \sum_{i=0}^{n} \binom{n}{i} k^i a_i. \]

The falling k-binomial transform F of a sequence A is the sequence F(A, k) = {fn}, where fn is given by

\[ f_n = \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i, \quad \text{if } k \neq 0; \]
\[ f_n = 0, \quad \text{if } k = 0. \]

The case k = 0 must be dealt with separately because 0 0 would occur in the formulas otherwise. Our definitions effectively take 0 0 to be 1. These turn out to be “good” definitions, in the sense that all the results discussed subsequently hold under our definitions for the k = 0 case. When k = 0, the k-binomial transform of A is the sequence \{a0, 0, 0, 0, \ldots\}, the rising k-binomial transform of A is \{a0, a0, a0, \ldots\}, and the falling k-binomial transform is the identity transform.

The k-binomial transform when k = 1/2 is of special interest; this is the binomial mean transform, sequence A075271. When k is a positive integer, these various versions of the binomial transform all have combinatorial interpretations similar to that of the binomial transform, although, unlike the binomial transform, they have a two-dimensional component.

If \( A \) represents the number of arrangements of n labeled objects with some property P, then wn represents the number of ways of dividing n objects such that

- In one dimension, the n objects are divided into two groups so that the first group has property P.
- In a second dimension, the n objects are divided into k labeled groups.

The interpretation of the second dimension could be something as simple as a coloring of each object from a choice of k colors, independent of the division of the objects in the first dimension. For example, if the input sequence is the derangement numbers, wn is the number of ways of dividing n labeled objects into two groups such that the objects in the first group are deranged and each of the n objects has been colored one of k colors, independently of the initial division into two groups.

With this interpretation of wn in mind, rn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and each object in the first group is further placed into one of k labeled groups (e.g., colored using one of k colors). Similarly, fn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and the objects in the second group are further placed into k labeled groups (e.g., colored using one of k colors).

Invariance of the Hankel Transform under the Binomial Transform

Layman proves that H(B(A)) = H(A) for any sequence A. He does this by showing that the Hankel matrix of order n of B(A) can be obtained by multiplying the Hankel matrix of order n of A by certain upper and lower triangular matrices, each of which have determinant 1.

We present a new proof of this result. Our proof technique suggests generalizations of the binomial transform, which we discuss in subsequent sections. We require the following lemma.

Lemma 1. Given a sequence A = \{a0, a1, a2, \ldots\}, create a triangle of numbers T using the following rule:
- The left diagonal of the triangle consists of the elements of A.
- Any number off the left diagonal is the sum of the number
to its left and the number diagonally above it to the left. Then
the sequence on the right diagonal is the binomial transform
of A. For example, the binomial transform of the
derangement numbers is the factorial numbers.
Figure 1 illustrates how the factorial numbers can be
generated from the derangement numbers using the triangle
described in Lemma 1. Although they do not use the term
binomial transform, Lemma 1 is essentially proven by
Graham, Knuth, and Patashnik.
We present a different proof, one that allows us to prove
similar results for the k-binomial transforms we discuss in
subsequent sections.

\[\begin{array}{cccccccc}
265 & 305 & 306 & 307 & 308 & 309 & 310 & 311 \\
302 & 303 & 304 & 305 & 306 & 307 & 308 & 309 \\
301 & 302 & 303 & 304 & 305 & 306 & 307 & 308 \\
300 & 301 & 302 & 303 & 304 & 305 & 306 & 307 \\
299 & 300 & 301 & 302 & 303 & 304 & 305 & 306 \\
298 & 299 & 300 & 301 & 302 & 303 & 304 & 305 \\
297 & 298 & 299 & 300 & 301 & 302 & 303 & 304 \\
296 & 297 & 298 & 299 & 300 & 301 & 302 & 303 \\
295 & 296 & 297 & 298 & 299 & 300 & 301 & 302 \\
294 & 295 & 296 & 297 & 298 & 299 & 300 & 301 \\
293 & 294 & 295 & 296 & 297 & 298 & 299 & 300 \\
292 & 293 & 294 & 295 & 296 & 297 & 298 & 299 \\
291 & 292 & 293 & 294 & 295 & 296 & 297 & 298 \\
290 & 291 & 292 & 293 & 294 & 295 & 296 & 297 \\
289 & 290 & 291 & 292 & 293 & 294 & 295 & 296 \\
288 & 289 & 290 & 291 & 292 & 293 & 294 & 295 \\
287 & 288 & 289 & 290 & 291 & 292 & 293 & 294 \\
286 & 287 & 288 & 289 & 290 & 291 & 292 & 293 \\
285 & 286 & 287 & 288 & 289 & 290 & 291 & 292 \\
\end{array}\]

Figure 1: Derangement triangle

Proof.
Let \( t_n \) be the \( n \)th element on the right diagonal of the triangle.
By construction of the triangle, we can see from Figure 2 that
the number of times element \( a_i \) contributes to the value of \( t_n \)
is the number of paths from \( a_i \) to \( t_n \). To move from \( a_i \) to \( t_n \)
requires \( n \) path segments, \( i \) of which move directly to the
right. Thus there are ways to choose which of the \( n \) ordered
segments are the rightward-moving segments, and the down
segments are completely determined by this choice.

\[\begin{array}{cccc}
a_0 & a_1 & a_2 & a_3 \\
a_2 & a_3 & a_4 & a_5 \\
a_4 & a_5 & a_6 & a_7 \\
a_6 & a_7 & a_8 & a_9 \\
a_8 & a_9 & a_{10} & a_{11} \\
a_{10} & a_{11} & a_{12} & a_{13} \\
a_{12} & a_{13} & a_{14} & a_{15} \\
a_{14} & a_{15} & a_{16} & a_{17} \\
a_{16} & a_{17} & a_{18} & a_{19} \\
a_{18} & a_{19} & a_{20} & a_{21} \\
a_{20} & a_{21} & a_{22} & a_{23} \\
a_{22} & a_{23} & a_{24} & a_{25} \\
a_{24} & a_{25} & a_{26} & a_{27} \\
a_{26} & a_{27} & a_{28} & a_{29} \\
a_{28} & a_{29} & a_{30} & a_{31} \\
\end{array}\]

Figure 2: Directed graph underlying the binomial transform

The binomial transform has the following combinatorial
interpretation: If \( a_n \) represents the number of arrangements
of \( n \) labeled objects with some property \( P \), then \( b_n \) represents
the number of ways of dividing \( n \) labeled objects into two
groups such that the first group has property \( P \).
In terms of the derangement and factorial numbers, then, as
\( D_n \) is the number of permutations of \( n \) ordered objects in
which no object remains in its original position, \( n! \) is the
number of ways that one can divide \( n \) labeled objects into
two groups, order the objects in the first group, and then
permute the first group objects so that none remains in its
original position.
The numbers in the triangle described in Lemma 1, not just
the right and left diagonals, can also have combinatorial
interpretations. For instance, the triangle of numbers in
Figure 1 is discussed as a combinatorial entity in its own
right in another article by the first author.
The number in row \( i \), position \( j \), in the triangle is the number
of permutations of \( i \) ordered objects such that every object
after \( j \) does not remain in its original position. We now give
our proof of Layman’s result.

Conclusion
We have introduced three generalizations of the binomial
transform: the \( k \)-binomial transform, the rising \( k \)-binomial
transform, and the falling \( k \)-binomial transform. We have
given a simple method for constructing these transforms, and
we have given combinatorial interpretations of each of them.
We have also shown how the generating function of a
sequence changes after applying one of the transforms.
This allows us to prove that several sequences in the On-Line
Encyclopedia of Integer Sequences are related by one of
these transforms, as well as prove three specific conjectures
in the OEIS. In addition, we have shown how the Hankel
transform of a sequence changes after applying one of the
transforms.
These results determine the Hankel transforms of several
sequences listed in the OEIS. We see several areas of further
study. One involves continuing to answer Layman’s
question. We have proved that the Hankel transform is
invariant under the falling \( k \)-binomial transform, and he
proves that the Hankel transform is invariant under the
binomial and invert transforms.

Theorem: (Layman) The Hankel transform is invariant under
the binomial transform.
Proof.
We define a procedure for transforming the Hankel matrix of
order \( n \) of a sequence \( A \) to the Hankel matrix of order \( n \) of
\( B(\text{A}) \) using only matrix row and column addition. While
perhaps more complicated than Layman’s proof, ours has the
virtue of being easily modified to give proofs for the Hankel
transforms of the \( k \)-binomial transforms that we discuss
subsequently.
The procedure is as follows:
1. Given a sequence \( A = \{ a_0, a_1, \ldots \} \), create the triangle of
numbers described in Lemma where \( T_{i,j} \) is the \(( i, j) \)th entry
in the triangle.
2. Let \( T_n \) be the following matrix consisting of numbers from
the left diagonal of \( T \):

\[\begin{bmatrix}
T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\
T_{1,0} & T_{2,0} & T_{3,0} & \cdots & T_{n+1,0} \\
T_{2,0} & T_{3,0} & T_{4,0} & \cdots & T_{n+2,0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{n,0} & T_{n+1,0} & T_{n+2,0} & \cdots & T_{2n,0} \\
\end{bmatrix}\]

Since \( a_i = T_{i,0} \), \( T_n \) is the Hankel matrix of order \( n \) of \( A \).
3. Then apply the following transformations to \( T_n \), where
rows and columns of the matrix are indexed beginning with
0. (a) Let \( i \) range from 1 to \( n \).
During stage \( i \), for each row \( j \geq i \), add row \( j - 1 \) to row \( j \)
and replace row \( j \) with the result. (b) Then let \( i \) again range from
1 to \( n \). During stage \( i \), for each column \( j \geq i \), add column \( j - 1 \)
to column \( j \) and replace column \( j \) with the result.
But this is the Hankel matrix of order \( n \) of \( B(\text{A}) \), as \( B(\text{A}) \)
is the right diagonal of triangle \( T \). Since the only matrix
manipulations we used were adding a row to another row and
adding a column to another column, and the determinant of a
matrix is invariant under these operations, the determinant of
the Hankel matrix of order \( n \) of \( A \) is equal to the determinant
of the Hankel matrix of order \( n \) of \( B(\text{A}) \).

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References