Some results on 3-Equitable prime cordial labeling of graphs

S Meena and A Archunan

Abstract

A 3-equitable prime cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to \{1, 2, 3, ..., $|V|$\} such that if an edge $uv$ is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and $gcd(f(u) + f(v), f(u) - f(v)) = 1$ the label 2 and if $gcd(f(u), f(v)) = 1$ and $gcd(f(u) + f(v), f(u) - f(v)) = 2$ and 0 otherwise, then the number of edges labeled with 1 and the number of edges labeled with 2 differ by atmost 1 for 0 $\leq i, j \leq 2$, if a graph has a 3-equitable prime cordial labeling then it is called a 3-equitable prime cordial graph. In this paper, we investigate the 3-equitable prime cordial labeling behavior of cycle with three chords and a cycle.

Keywords: 3-equitable prime cordial labeling, 3-equitable prime cordial graph

Introduction

The graphs considered here are finite, connected, undirected and simple. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. For various graph theoretic notations and terminology we follow Harary \cite{1} and for number theory we follow Burton \cite{2}. We will give the brief summary of definitions which are useful for the present investigations.

Definition 1: If the vertices of the graph are assigned values subject to certain conditions it is known as graph labeling.

Most interesting graph labeling problems have three important characteristics.
1. A set of numbers from which vertex labels are chosen.
2. A rule that assigns a value to each edge.
3. A condition that these values must satisfy.

For detailed survey on graph labeling one can refer Gallian \cite{3}. Vast amount of literature is available on different types of graph labeling. According to Beineke and Hegde \cite{4} graph labeling serves as a frontier between number theory and structure of graphs. Labeled graphs have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A detailed study of variety of applications of graph labeling is given by Bloom and Golomb \cite{5}.

The present work is to aimed to discuss one such labeling known as 3-equitable prime cordial labeling.

Definition 2: Let $G = (V(G), E(G))$ be a graph. A mapping $f: V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

For and edge $e = uv$, the induced edge labeling $f^* = E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of $G$ having labels of 0 and 1 respectively under $f$ and $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under $f^*$. 
Let odd except \( h(1) \equiv 1 \). A graph \( G \) is cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [6]. Many researchers have studied cordiality of graphs. Cahit [6] proved that tree is cordial and \( K_n \) is cordial if and only if \( n \leq 3 \). Vaidya et al. [7] have also discussed the cordiality of various graphs.

Definition 4: Let \( G = (V,E) \) be a graph. A mapping \( f:V(G) \rightarrow \{0,1,2\} \) is called ternary vertex labeling of \( G \) and \( f(v) \) is called the label of the vertex \( v \) of \( G \) under \( f \).

For an edge \( e = uv \), the induced edge labeling is given \( f^*(e) = |f(u) - f(v)| \). Let \( v_f(0), v_f(1), v_f(2) \) be the number of vertices of \( G \) having labeling 0,1,2 respectively under \( f \) and \( e_f(0), e_f(1), e_f(2) \) be the number of edges having labels 0,1,2 respectively under \( f^* \).

Definition 5: A ternary vertex labeling of a graph \( G \) is called a 3-equitable labeling if \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for all \( 0 \leq i,j \leq 2 \). A graph \( G \) is 3-equitable if it admits 3-equitable labeling. The concept of 3-equitable labeling was introduced by Cahit [8]. Many researchers have studied 3-equitability of graphs.

Definition 6: A prime cordial labeling of a graph \( G \) with vertex set \( V(G) \) is a bijection defined by \( f:V(G) \rightarrow \{1,2,3,\ldots,|V|\} \)

defined by \( f(e = uv) = 1 \) if \( \text{gcd}(f(u), f(v)) = 1 \) and \( 0 \) otherwise

and \( |e_f(i) - e_f(j)| \leq 1 \). A graph which admits prime cordial labeling is called a prime cordial graph.

The concept of prime cordial labeling was introduced by Sundaram et al. [10] and in the same paper they investigate several results on prime cordial labeling. Vaidya and Vihol [11] have also discussed prime cordial labeling in the context of graph operations.

By combining the concepts of prime cordial labeling and 3-equitable labeling and by using the result in Number Theory that \( \text{gcd}(a,b) = 1 \), then \( \text{gcd}(a + b, a - b) = 1 \) or 2, we introduce a new concept called 3-equitable prime cordial labeling as follows.

Definition 7: A 3-equitable prime cordial labeling of a graph \( G \) with vertex set \( V(G) \) is a bijection \( f:V(G) \rightarrow \{1,2,3,\ldots,p\} \) defined by

\[
f(e = uv) = \begin{cases} 
1 & \text{if } \text{gcd}(f(u), f(v)) = 1 \text{ and } \text{gcd}(f(u) + f(v), f(u) - f(v)) = 1 \\
2 & \text{if } \text{gcd}(f(u), f(v)) = 1 \text{ and } \text{gcd}(f(u) + f(v), f(u) - f(v)) = 2 \\
0 & \text{otherwise}
\end{cases}
\]

And \( |e_f(i) - e_f(j)| \leq 1 \) for all \( 0 \leq i,j \leq 2 \). A graph which admits 3-equitable prime cordial graph labeling is called a 3-equitable prime cordial graph.

Definition 8: A chord of cycle \( C_n \) is an edge joining two non-adjacent vertices of \( C_n \).

Theorem 1
Cycle \( C_n \) with three chord is 3-equitable prime cordial if \( n \) is odd except \( n \equiv 1(\text{mod}3) \) \( (n \geq 9) \)

Proof
Let \( G \) be a cycle \( C_n \) with three chords.
Let \( u_1, u_2, u_3 \) be the vertices of cycle \( C_n \) and let \( e = u_1u_3 \) be one chord of cycle \( C_n \), \( e = u_3u_n \) and \( e = u_1u_{n-2} \) be the other two chords of \( C_n \). Let \( f(u_1) = 2, f(u_2) = 4, f(u_3) = 6 \) and \( f(u_{n-2}) = n \) put \( f(u_3) = n \) Label the next three vertices is \( u_4, u_5, u_6 \) by consecutive odd numbers 1, 2, 3 respectively, again the next three vertices \( u_7, u_8, u_9 \) by consecutive even numbers 8, 10, 12 respectively. Repeat the process until all the vertices get labeled of there does not exist three even numbers (or) odd numbers then according to existence of number give the label.

Illustration: The 3-equitable prime cordial labeling of cycle \( C_9 \) with three chords is as shown below.
Case (ii)
\( n \equiv 1 \pmod{3} \)
Here one chord is \( u_1u_3 \), and the other two chords are \( u_3u_n, u_4u_{n-1} \)

Case (iii)
\( n \equiv 2 \pmod{3} \)
Here one chord is \( u_1u_3 \), and the other two chords are \( u_4u_n, u_4u_{n-2} \)

Case (iv)
\( n \equiv 0 \pmod{3} \) \( n \equiv 0 \pmod{10} \)
Here one chord is \( u_1u_3 \), and the other two chords are \( u_3u_{n-1}, u_4u_4 \).

Theorem 3
\( G_n \) with two pendent vertices is equitable prime cordial. \( n \) is even \( n \geq 10 \).

Proof
Let \( G \) be a graph. Let \( v_1, v_2, v_3, \ldots, v_n \) be the vertices of the cycle.
Let \( f(v_1) = 1, f(v_2) = 3, f(v_3) = 5 \) and \( f(v_n) = n \)
Label the next three vertices \( v_4, v_5, v_6 \) by consecutive odd numbers 2, 4, 6 respectively. Again the next three vertices \( v_7, v_8, v_9 \) by consecutive even numbers 7, 9, 11 respectively. Repeat the problem until all the vertices get labeled. If there does not exist three even numbers (or) odd numbers then according to the existence of numbers give the label.
Here interchanging the labeling of the vertices. 
\((25+k \ 30)\) by \((25+k \ 30) + 2, \ k = 0, 1, 2, 3, \ldots \)

Conclusion
By using a property from number theory, we have introduced 3-equitable prime cordial labeling of graphs. In the present work, we investigated 3-equitable prime cordial labeling behavior of standard graphs only. To investigate analogous results for different graphs as well as in the context of various graph labeling problems is an open area of research.

References

Testing the diagrams and figures in the text for alignment and readability.