Minimal pgrω-Open sets and Maximal pgrω-Closed sets in Topological spaces

R. S. Wali, Vijaykumari T. Chilakwad

Abstract
In this paper the new sets called minimal pgrω-open sets and maximal pgrω-closed sets in a topological space are introduced which are the pgrω-open sets and pgrω-closed sets respectively. The complement of minimal pgrω-open set is a maximal pgrω-closed set. Some properties of the maximal-semi-pgrω-open sets, minimal-semi-pgrω-closed sets, Minimal pgrω-continuous maps, Maximal pgrω-continuous maps and Tmin−pgrw, Tmax−pgrw, Min-Tpgrw, Max-Tpgrw-Spaces are studied.

Keywords: Minimal pgrω-open set, Maximal pgrω-closed set, Tmin−pgrw, Tmax−pgrw, Min-Tpgrw, Max-Tpgrw –Spaces.

1. Introduction

Definition 1.1: [1] A non-empty open proper subset U of a topological space X is said to be a minimal open set if any open set which is contained in U is either ϕ or U.

Definition 1.2: [2] A non-empty open proper subset U of a topological space X is said to be a maximal open set if any open set which contains U is either X or U.

Definition 1.3: [3] A non-empty closed proper subset F of a topological space X is said to be a minimal closed set if any closed set which is contained in F is ϕ or F.

Definition 1.4: [3] A non-empty closed proper subset F of a topological space X is said to be a maximal closed set if any closed set which contains F is either X or F.

Definition 1.5: [5] A subset A of a topological space X is called a pgrω-closed set if pcl(A) ⊆ U whenever A ⊆ U and U is rω-open in X.

Definition 1.6: [5] A subset A of a topological space X is called a pgrω-open set in X if Ac is a pgrω-closed set in X.

Definition 1.7: A co-finite subset of a set X is a subset whose complement in X is a finite set.

Definition 1.8: [6] A map f: (X, T1)→(Y, T2) is called a pre generalised regular weakly-continuous map (pgrw-continuous map) if the inverse image f⁻¹(V) of every closed set V in Y is pgrw-closed in X.

2. Minimal pgrω-open sets
Definition 2.1: A non-empty proper pgrω-open subset U of a topological space X is said to be a minimal pgrω-open set if any pgrω-open set contained in U is either ϕ or U.

Note: X and ϕ are pgrw-open, but not minimal pgrw-open sets.
**Example 2.2:** Let $X = \{a, b, c, d\}$ and topology on $X$ be $T=\{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}\}$.
Minimal open sets are $\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}$. 
Minimal pgrw-open sets are $\{a\}, \{b\}, \{c\}$. Here $\{a\}$ is open, but not minimal pgrw-open set. 
Minimal pgrw-open sets are $\{a\}, \{b\}, \{c\}$. 

**Theorem 2.5:** Every non-empty finite pgrw-open set contains at least one minimal pgrw-open set.

**Proof:** Let V be a non-empty finite pgrw-open set. If V is a minimal pgrw-open set, then the statement holds true. If V is not a minimal pgrw-open set, then there exists an open set $V_1$ such that $\emptyset \neq V_1 \subset V$. If $V_1$ is a minimal pgrw-open set, then the statement holds true. If V is not a minimal pgrw-open set, then there exists an open set $V_2$ such that $\emptyset \neq V_2 \subset V$. Continuing this process we have a sequence of pgrw-open sets $V_k \subset V_{k+1} \subset \cdots \subset V_n \subset V$. Since V is a finite set, this process repeats only finitely and so finally we get a minimal pgrw-open set $V_n$ for some positive integer n such that $V_n \subset V$.

**Corollary 2.6** If V be a finite minimal open set, then there exists at least one minimal pgrw-open set U such that $U \subset V$.

**Proof:** If V is a finite minimal open set, then V is a non-empty finite pgrw-open set and so by theorem 2.5 there exists at least one minimal pgrw-open set U such that $U \subset V$.

3. **Maximal pgrw-closed sets:**

**Definition 3.1:** A non-empty proper pgrw-closed subset F of a topological space X is said to be a maximal pgrw-closed set if any pgrw-closed set which contains F is either X or F.

**Note:** X and $\emptyset$ are pgrw-closed, but not maximal pgrw-closed.

**Example 3.2:** Let $X = \{a, b, c, d\}$ and topology on $X$ be $T=\{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}\}$. 
Closed sets are $X, \emptyset, \{b\}, \{a, b\}, \{a, c\}, \{c, d\}, \{d\}$. 
Maximal closed sets are $\{b\}, \{c\}, \{d\}$. 
Pgrw-closed sets are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}$. 
Maximal pgrw-closed sets are $\{b\}, \{c\}, \{a, c\}, \{a, d\}$.

**Example 3.3:** Let $X = \{a, b, c, d\}$, $\emptyset \neq V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots \subset X-F$. If $V_1$ is a minimal pgrw-open set, then there exists a pgrw-open set $V_2$ such that $\emptyset \neq V_2 \subset V_1$. Continuing this process we have a sequence of pgrw-open sets $V_k \subset V_{k+1}$ such that $\emptyset \neq V_k \subset \cdots \subset V_n \subset \cdots \subset X-F$. Therefore X-F is a non-empty finite pgrw-open set.

**Theorem 3.5:** A non-empty proper subset F of a topological space X is a maximal pgrw-closed set iff X-F is a minimal pgrw-open set.

**Proof:** Let F be a maximal pgrw-closed set. Suppose X-F is not a minimal pgrw-open set. Then there exists a pgrw-open set $U \neq X-F$ such that $\emptyset \neq U \subset X-F$ that is proper $\emptyset \neq X-U \subset X-F$ is a pgrw-closed set. This contradicts our assumption that F is a Maximal pgrw-closed set. Conversely, suppose X-F is a minimal pgrw-open set and F is not a maximal pgrw-closed set. Then there exists a pgrw-closed set $E \subset X-F$ such that $\emptyset \neq E \subset X-F$ and $X-E$ is a pgrw-open set. This contradicts our assumption that X-F is a minimal pgrw-open set. Therefore F is a Maximal pgrw-closed set.

**Theorem 3.6:** If F is a non-empty proper co-finite pgrw-closed subset of a topological space X, then there exists a co-finite maximal pgrw-closed set E such that $F \subset E$.

**Proof:** Let F be a non-empty proper co-finite pgrw-closed set in a topological space X. 
$\Rightarrow$ X-F is a non-empty finite pgrw-open set. 
$\Rightarrow$ there exists a finite maximal pgrw-open set U such that $U \subset X-F$ by th 2.5 
$\Rightarrow$ there exists a co-finite Maximal closed set E= X-U such that $E \subset X-F$.

**Definition 3.7:** A non-empty proper pgrw-open subset A of a topological space X is said to be a maximal pgrw-open set if any pgrw-open set containing A is either A or X.

**Definition 3.8:** A non-empty proper pgrw-closed subset A of a topological space X is said to be a minimal pgrw-closed if any pgrw-closed set contained in A is either $\emptyset$ or A.
Example 3.9. Let $X = \{a, b, c, d\}$, $T = \{X, \emptyset, \{a\}, \{c, b\}, \{a, c, d\}\}$. Pgrw-open sets are $X, \emptyset, \{a\}, \{c, b\}, \{a, c, d\}$. Pgrw-closed sets are $\{b\}, \{c\}, \{d\}$. Maximal pgrw-closed sets are $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}$. Minimal pgrw-closed sets are $\emptyset, \{b\}, \{c\}, \{d\}$.

4. Maximal-semi-pgrw-open sets and Minimal-semi-pgrw-closed sets:

Definition 4.1. A subset $A$ in a topological space $X$ is said to be maximal-semi-pgrw-open if there exists a maximal-pgrw-open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$.

Definition 4.2. A subset $A$ in a topological space $X$ is said to be minimal-semi-pgrw-closed if $X - A$ is maximal-semi-pgrw-open set of $X$.

Remark 4.3. Every maximal pgrw-open (minimal pgrw-closed) set is a maximal-semi-pgrw-open (minimal-semi-pgrw-closed) set.

Theorem 4.4. If $A$ is a maximal-semi-pgrw-open set of a topological space $X$ and $A \subseteq B \subseteq \text{cl}(A)$, then $B$ is a maximal-semi-pgrw-open set of $X$.

Proof: If $A$ is a maximal-semi-pgrw-open set of $X$, then there exists a maximal-pgrw-open set $U$ of $X$ such that $U \subseteq A \subseteq \text{cl}(U)$ and if $A \subseteq B \subseteq \text{cl}(A)$, then $U \subseteq A \subseteq \text{cl}(A) \subseteq \text{cl}(B)$. Thus $B$ is a maximal-semi-pgrw-open set of $X$.

Theorem 4.5. A minimal-semi-pgrw-closed subset in a topological space $X$ if and only if there exists a minimal-pgrw-closed set $B$ in $X$ such that $\text{int}(B) \subseteq A \subseteq B$.

Proof: Suppose $A$ is a minimal-semi-pgrw-closed set in a topological space $X$. Then by definition $X - A$ is maximal-semi-pgrw-open set of $X$. There exists a maximal pgrw-open set $U$ of $X$ such that $U \subseteq X - A \subseteq \text{cl}(U)$. That is $\text{int}(X - U) = X - \text{cl}(U) \subseteq A \subseteq X - U$. Let $B = X - U$, so that $B$ is a minimal pgrw-closed set in $X$ such that $\text{int}(B) \subseteq A \subseteq B$. Conversely Suppose that there exists a minimal pgrw-closed set $B$ in $X$ such that $\text{int}(B) \subseteq A \subseteq B$. Then $X - B \subseteq X - A \subseteq X - \text{int}(B) = \text{cl}(X - B)$. That is there exists a maximal pgrw-open set $U = X - B$ such that $U \subseteq X - A \subseteq \text{cl}(U)$. This implies $X - A$ is maximal-semi- gprw-open in $X$. Hence $A$ is minimal-semi-pgrw-closed in $X$.

Theorem 4.6. If $A$ and $B$ are subsets of a topological space $X$ such that $B$ is minimal-semi-pgrw-closed and $\text{int}(B) \subseteq A \subseteq B$, then $A$ is also minimal-semi-pgrw-closed in $X$.

Proof: Let $B$ be a minimal-semi-pgrw-closed set of $X$. Then by th. 4.5 there exists a minimal pgrw-closed set $U$ such that $\text{int}(U) \subseteq B \subseteq U$. Since $\text{int}(B) \subseteq A \subseteq B$, we have $\text{int}(U) \subseteq \text{int}(B) \subseteq A \subseteq B \subseteq U$. That is $\text{int}(U) \subseteq A \subseteq U$. Therefore $A$ is a minimal-semi-pgrw-closed set in $X$.

Theorem 4.7. $Y$ is any open subspace of a topological space $X$ and $A \subseteq Y$. If $A$ is a maximal-semi-pgrw-open set of $X$, then $A$ is also a maximal-semi-pgrw-open set of $Y$.
\( f^{-1}(\{c,d\}) = \{b,c\} \) and is pgrw-closed in \( X \). So \( f \) is minimal pgrw-continuous. \( \{b,c,d\} \) is closed in \( Y \). \( f^{-1}(\{b,c,d\}) = \{a,b,c\} \) is not pgrw-closed. Hence \( f \) is not pgrw-continuous.

**Theorem 5.6.** \( X \) and \( Y \) are two topological spaces. A map \( f: X \to Y \) is minimal (maximal) pgrw-continuous if and only if the inverse image of every maximal (minimal) open set in \( Y \) is a pgrw-open set in \( X \).

**Proof.** Let \( A \) be a maximal (minimal) open set in \( Y \). Then \( A^c \) is minimal (maximal) closed in \( Y \). As \( f: X \to Y \) is minimal (maximal) pgrw-continuous, \( f^{-1}(A) = f^{-1}(A^c) \) is pgrw-closed in \( X \). Hence \( f^{-1}(A) \) is maximal (minimal) pgrw-closed set in \( X \).

Conversely suppose \( f: X \to Y \) is such that inverse image of every maximal (minimal) open set in \( Y \) is a pgrw-open set in \( X \). Let \( A \) be a minimal (maximal) closed set in \( Y \). Then \( A^c \) is maximal (minimal) open set in \( Y \). But \( f^{-1}(A) = f^{-1}(A^c) \). Therefore \( f^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( X \).

**Theorem 5.7.** \( X \) and \( Y \) are two topological spaces and \( A \) is a nonempty subset of \( X \). If \( f: X \to Y \) is a minimal (maximal) pgrw-continuous map, then the restriction map \( f_\text{A}: A \to Y \) is minimal (maximal) pgrw-continuous.

**Proof.** Let \( f: X \to Y \) be a minimal (maximal) pgrw-continuous map. Let \( B \) be any minimal (maximal) closed set in \( Y \). Since \( f \) is minimal (maximal) pgrw-continuous, \( f^{-1}(B) = f^{-1}(B)^c \) is a pgrw-closed set in \( X \). Hence \( f^{-1}(B) \) is minimal (maximal) pgrw-closed set in \( X \).

Converse is not true.

**Example 5.9.** \( X = \{a,b,c,d\}, T_1 = \{X, \varnothing, \{a,b\}, \{c,d\}\} \)
\( Y = \{a,b,c\}, T_2 = \{Y, \varnothing, \{a\}, \{a,c\}\} \)
Pgrw-closed sets in \( X \) are all subsets of \( X \). Minimal pgrw-closed sets in \( X \) are \( \{a\}, \{b\}, \{c\} \).
\( \{b\} \) is the only minimal closed set in \( Y \). Define \( f: X \to Y \) by \( f(a) = b, f(b) = b, f(c) = a, f(d) = c \).
\( f^{-1}(\{b\}) = \{a,b\} \) is pgrw-closed in \( X \). Hence \( f \) is minimal pgrw-continuous.
\( \{a,b\} \) is not minimal pgrw-closed. \( \{a\} \) is not minimal pgrw-closed. \( \{b\} \) is not minimal pgrw-closed. \( \{c\} \) is not minimal pgrw-closed. \( \{d\} \) is not minimal pgrw-closed.

\( f \) is not minimal pgrw-irresolute.

**Theorem 5.10.** \( X \) and \( Y \) are two topological spaces. A map \( f: X \to Y \) is minimal maximal (maximal minimal) pgrw-continuous if and only if the inverse image of each maximal (minimal) open set in \( Y \) is a pgrw-open set in \( X \).

**Proof.** Suppose \( f: X \to Y \) is minimal maximal (maximal minimal) pgrw-continuous. Let \( A \) be a maximal (minimal) open set in \( Y \). Then \( A^c \) is a minimal (maximal) closed set in \( Y \). As \( f \) is minimal maximal (maximal minimal) pgrw-continuous, \( f^{-1}(A) = f^{-1}(A^c) \) is a minimal (maximal) pgrw-open set in \( X \).

Conversely suppose \( f: X \to Y \) is such that inverse image of each maximal (minimal) open set in \( Y \) is a pgrw-open set in \( X \).

**Theorem 5.11.** Every minimal maximal (maximal minimal) pgrw-continuous map is a minimal (maximal) pgrw-continuous map.

**Proof.** Let \( f: X \to Y \) be a minimal maximal (maximal minimal) pgrw-continuous map. Let \( A \) be any minimal (maximal) closed set in \( Y \). Since \( f \) is minimal maximal (maximal minimal) pgrw-continuous, \( f^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( X \). Therefore \( f \) is minimal (maximal) pgrw-continuous.

**Definition 5.12.** A topological space \((X,T)\) is called a

(i) \( T_{\min-pgrw} \) space if every pgrw-closed set in it is minimal closed.

(ii) \( T_{\max-pgrw} \) space if every pgrw-closed set in it is maximal closed.

(iii) \( Min-T_{pgrw} \) Space if every minimal pgrw-closed set in it is minimal closed.

(iv) \( Max-T_{pgrw} \) Space if every maximal pgrw-closed set in it is maximal closed.

**Theorem 5.13.** If \( f: X \to Y \) is a minimal (maximal) pgrw-continuous map and \( Y \) is a \( T_{\min-pgrw} \) (\( T_{\max-pgrw} \)) space, then \( f \) is pgrw-continuous.

**Proof.** Let \( f: X \to Y \) be a minimal (maximal) pgrw-continuous function. Let \( A \) be a closed set in \( Y \). Since every closed set is pgrw-closed, \( A \) is pgrw-closed. By hypothesis \( Y \) is a \( T_{\min-pgrw} \) (\( T_{\max-pgrw} \)) space, it follows that \( A \) is a minimal (maximal) pgrw-closed set in \( Y \). Since \( f \) is minimal (maximal) pgrw-continuous, \( f^{-1}(A) \) is pgrw-closed in \( X \). Therefore \( f \) is pgrw-continuous.
minimal)-pgrw-continuous map, \( f^{-1}(A) \) is a maximal pgrw-closed set in \( X \). So \( f^{-1}(A) \) is pgrw-closed. Hence \( f: X \to Y \) is pgrw-continuous.

**Theorem 5.16.** If \( f: X \to Y \) and \( g: Y \to Z \) are minimal (maximal) pgrw-irresolute maps and \( Y \) is a Min-T_{pgrw}(Max-T_{pgrw}) space, then \( g0f: X \to Z \) is a minimal (maximal) pgrw-irresolute map.

**Proof.** Let \( A \) be any minimal (maximal) closed set in \( Z \). Since \( g \) is minimal (maximal) pgrw-irresolute, \( g^{-1}(A) \) is a minimal (maximal) pgrw-closed set in \( Y \). Since \( Y \) is a Min-T_{pgrw}(Max-T_{pgrw}) space, \( g^{-1}(A) \) is a minimal (maximal) closed set in \( Y \). Again since \( f \) is minimal (maximal) pgrw-irresolute, \( f^{-1}(g^{-1}(A)) = (g0f)^{-1}(A) \) is minimal (maximal) pgrw-closed set in \( X \). Therefore \( g0f \) is a minimal (maximal) pgrw-irresolute.

**Theorem 5.17.** If \( f: X \to Y \) is maximal (minimal) pgrw-irresolute, \( Y \) is a Max-T_{pgrw}(Min-T_{pgrw}) space and \( g: Y \to Z \) is minimal maximal (maximal minimal)-pgrw-continuous, then \( g0f: X \to Z \) is a minimal maximal (maximal minimal) pgrw-continuous map.

**Proof:** Let \( A \) be any minimal (maximal) closed set in \( Z \). Since \( g \) is minimal maximal (maximal minimal) pgrw-continuous, \( g^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( Y \). Since \( Y \) is a Max-T_{pgrw}(Min-T_{pgrw}) space, \( g^{-1}(A) \) is a maximal (minimal) closed set. Again since \( f \) is maximal (minimal) pgrw-irresolute, \( f^{-1}(g^{-1}(A)) = (g0f)^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( X \). Hence \( g0f \) is a minimal maximal (maximal minimal)-pgrw-continuous map.

**Theorem 5.18.** If \( f: X \to Y \) and \( g: Y \to Z \) are minimal maximal (maximal minimal)-pgrw-continuous maps and if \( Y \) is a T_{min-pgrw}(T_{max-pgrw}) space, then \( g0f: X \to Z \) is a minimal maximal (maximal minimal) pgrw-continuous map.

**Proof:** Let \( A \) be any minimal (maximal) closed set in \( Z \). Since \( g \) is minimal maximal (maximal minimal) pgrw-continuous, \( g^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( Y \) and so \( g^{-1}(A) \) is a pgrw-closed subset of \( Y \). Since \( Y \) is T_{min-pgrw}(T_{max-pgrw}) space, \( g^{-1}(A) \) is a minimal (maximal) closed set in \( Y \). Again since \( f \) is minimal maximal (maximal minimal) pgrw-continuous, \( f^{-1}(g^{-1}(A)) = (g0f)^{-1}(A) \) is a maximal (minimal) pgrw-closed set in \( X \). Hence \( g0f \) is a minimal maximal (maximal minimal) pgrw-continuous map.

6. References