On $\beta^*g$ closed sets in Topological Spaces

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Abstract
The aim of this paper is to introduce the concept of $\beta^*g$ closed and open set and its closure and interior. Some of the fundamental properties of this set are studied. And some of their properties are also given.

Keywords: g closed set, $\beta g$ closed set, $\beta g$ open set, $\beta$ open.

1. Introduction

2. Preliminaries
Definition 2.1: A subset $A$ of a topological space $(X, \tau)$ is called $\beta$ open [1] if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$, whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.2: A subset $A$ of $X$ is called pre-open set [9] if $A \subseteq \text{int}(\text{cl}(A))$ and pre-closed [7] set if $A \subseteq \text{cl}(\text{int}(A))$.

Definition 2.3: A subset $A$ of $X$ is called $\alpha$-open [10] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $\alpha$-closed [11] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

Definition 2.4: A subset $A$ of $X$ is called 0-closed [15] if $A = \text{cl}(\emptyset(A))$, where $\emptyset(A) = \{x \in X : cl(U) \cap A \neq U \in A\}$.

Definition 2.5: A subset $A$ of $X$ is called $\delta$-closed [15] if $A = \text{cl}(\emptyset(A))$, where $\emptyset(A) = \{x \in X : cl(U) \cap A \neq U \in A\}$.

Definition 2.6: A subset $A$ of $X$ is called regular open (briefly r-open) [8] set if $A = \text{int}(\text{cl}(A))$ and regular closed (briefly r-closed) [6] set if $A = \text{cl}(\text{int}(A))$.

Definition 2.7: A subset $A$ of $X$ is called generalized closed (briefly g-closed) [6] set if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 2.8: A subset $A$ of $X$ is called Semi-generalized closed (briefly sg-closed) [2] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open in $X$.

Definition 2.9: A subset $A$ of $X$ is called Generalized $\alpha$-closed (briefly $g\alpha$-closed) [3] if $\alpha$-$\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

Definition 2.10: A subset $A$ of $X$ is called Generalized semi-pre closed (briefly gsp-closed) [10] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

Definition 2.11: A subset $A$ of $X$ is called Regular generalized closed (briefly rg-closed) [11] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$. 
Definition 2.12: A subset A of X is called $\theta$-generalized closed (briefly $\theta$-g-closed) \([5]\) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$.

Definition 2.13: A subset A of X is called $\delta$-generalized closed (briefly $\delta$-g-closed) \([13]\) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$.

Definition 2.14: A subset A of X is called Strongly generalized closed (briefly g*-closed) \([9]\) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ in X.

Definition 2.15: A subset A of X is called Weakly generalized closed (briefly wg-closed) \([8]\) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

Definition 2.16: A subset A of X is called Semi weakly generalized closed (briefly swg-closed) \([8]\) if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and semi open in X.

3. On $\beta g^*$ closed set

Definition 3.1: A subset A of a topological space $(X,\tau)$ is called $\beta g^*$-closed set if $\text{gcl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\beta$ open in X.

Remark 3.2: $\emptyset$ and X are $\beta g^*$-closed subset of X.

Remark 3.3: (i) Every g closed set is $\beta g^*$ closed set, since whenever $A \subseteq U$, $\text{gcl}(A) \subseteq U$.

(ii) Every $g^*$ closed set is g closed. Therefore every $g^*$ closed set is $\beta g^*$-closed set.

Theorem 3.4: Every closed set is $\beta g^*$-closed set but not conversely.

Proof: Let A be a closed set such that $A \subseteq U$ and $U$ is $\beta$ open. Therefore $A=\text{cl}(A)$ and this implies $\text{gcl}(A) \subseteq U$. Hence $\text{gcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\beta$-open. Therefore A is a $\beta g^*$ closed set.

Example 3.5: Let $X=\{a, b, c, d\}$, $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Here $A=\{a, d\}$ is $\beta g^*$ closed but not closed set in X.

Remarks 3.6: (i) Every $\theta$-closed set is a closed set. Therefore every $\theta$-closed set is $\beta g^*$ closed.

(ii) Every $\pi$ closed is closed. Therefore every $\pi$ closed is $\beta g^*$ closed.

(iii) Every $\delta$ closed is closed. Therefore every $\delta$ closed is $\beta g^*$ closed.

Theorem 3.7: Every regular closed set is $\beta g^*$ closed but not conversely.

Proof: Let A be a regular closed set, such that $A \subseteq U$ and U is $\beta$-open. Every regular closed set is closed. By Theorem 3.3, A is $\beta g^*$ closed set.

Example 3.8: Let $X=\{a, b, c, d\}$, $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $A=\{d\}$ is $\beta g^*$ closed but not regular closed.

Theorem 3.9: Every gr closed set is $\beta g^*$ closed.

Proof: Let A be a gr closed set such that $A \subseteq U$ and U is $\beta$-open. Every generalized regular closed set is closed. By Theorem 3.4, every gr closed set is a $\beta g^*$ closed set.

Theorem 3.10: Every w-closed set is $\beta g^*$ closed set, but not conversely.

Proof: Let A be a w-closed set. Therefore $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is semi open. Then $\text{gcl}(A) \subseteq U$. Therefore $\text{gcl}(A) \subseteq U$, whenever $A \subseteq U$ and U is semi open. Since every semi open set is $\beta$-open. Therefore every w-closed set is $\beta g^*$ closed set.

Example 3.11: Let $X=\{a, b, c, d\}$, $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Here $A=\{b, d\}$ is $\beta g^*$ closed but not a closed set in X.

Remark 3.12: The Union of two $\beta g^*$ closed subsets of X is also $\beta g^*$ closed subset of X.

Proof: Assume that A and B are $\beta g^*$ closed sets in X, such that $A \cup B \subseteq U$ and U is $\beta$ open. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are $\beta g^*$ closed set. Therefore $\text{gcl}(A) \subseteq U$ and $\text{gcl}(B) \subseteq U$. Hence $\text{gcl}(A \cup B) = \text{gcl}(A) \cup \text{gcl}(B) \subseteq U$. That is $A \cup B$ is $\beta g^*$ closed set.

Theorem 3.13: Let $A \subseteq B \subseteq \text{cl}(A)$ and A is a $\beta g^*$ closed subset of $(X, \tau)$ then B is also a $\beta g^*$ closed subset of $(X, \tau)$.

Proof: Since A is a $\beta g^*$ closed subset of $(X, \tau)$, $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$, U is $\beta$ open subset of X. Let $A \subseteq B \subseteq \text{cl}(A)$. Therefore $\text{cl}(A) = \text{cl}(B)$. Let V be a open subset of X such that $A \subseteq V$. So A is $\beta g^*$ closed subset of X, $\text{cl}(A) \subseteq V$. That is $\text{gcl}(B) \subseteq V$. Hence B is also a $\beta g^*$ closed subset of X.

Theorem 3.14: Let $A \subseteq B \subseteq \text{cl}(A)$, $B \subseteq \text{cl}(A \cap C)$ and $A$ is a $\beta g^*$ closed subset of $(X, \tau)$.

Proof: Let A be a $\beta g^*$ closed subset of X. Also let M be closed subset of X such that $M \subseteq \text{cl}(A \cap C)$ and $M \subseteq \text{cl}(A)$. Hence $M \subseteq \text{cl}(A \cap C)$. Therefore $M \subseteq \text{cl}(A \cap C)$. Hence $M \subseteq \text{cl}(A \cap C)$ and $M \subseteq \text{cl}(A \cap C)$. Hence $M \subseteq \text{cl}(A \cap C)$.

Theorem 3.15: A subset A of X is $\beta g^*$-closed set iff $\text{gcl}(A) \cap C$ contains no non-empty closed set in X.

Proof: Let A be a $\beta g^*$ closed subset of X. Also let M be closed subset of X such that $M \subseteq \text{cl}(A \cap C)$. Therefore $M \subseteq \text{cl}(A \cap C)$. Hence $M \subseteq \text{cl}(A \cap C)$ and $M \subseteq \text{cl}(A \cap C)$. Hence $M \subseteq \text{cl}(A \cap C)$. Therefore $M \subseteq \text{cl}(A \cap C)$.

Theorem 3.16: A subset A of X is a $\beta g^*$ closed set in X iff $\text{gcl}(A) \cap C$ contains no non-empty $\beta$ closed set in X.

Proof: Suppose that F is a non-empty $\beta$ closed subset of $\text{gcl}(A) \cap C$. Now $F \subseteq \text{cl}(A \cap C)$. Then $F \subseteq \text{cl}(A \cap C)$. Therefore $F \subseteq \text{cl}(A \cap C)$. Hence $F \subseteq \text{gcl}(A \cap C)$. So $F \subseteq \text{gcl}(A \cap C)$. Therefore $F \subseteq \text{gcl}(A \cap C)$. Hence $F \subseteq \text{gcl}(A \cap C)$. Therefore $F \subseteq \text{gcl}(A \cap C)$.

Suppose that $\text{gcl}(A) \subseteq U$ is $\beta$ open. Suppose that $\text{gcl}(A) \cap C$ contains no non-empty $\beta$ closed set. Conversely assume that $\text{gcl}(A) \cap C$ contains no non-empty $\beta$ closed set. Let A is a $\beta g^*$ closed set and contained in $\text{gcl}(A) \cap C$. Which is a contradiction. Therefore $\text{gcl}(A) \subseteq U$ and hence A is a $\beta g^*$ closed set.

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Example 3.17: The figure 1 justified with the following
examples.
Let $X=\{a, b, c, d\}$, be with the topology $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then
1. Closed sets in $X$ are $X, \emptyset, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$
2. $\beta g^*$ closed sets in $X$ are $X, \emptyset, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
3. $\alpha$ closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$
4. Pre closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$
5. Semi closed sets in $X$ are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
6. $\beta g^*$ closed sets in $X$ are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
7. Regular closed sets in $X$ are $X, \emptyset, \{a, c, d\}, \{b, c, d\}$
8. $g$ closed sets in $X$ are $X, \emptyset, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
9. $g^*$ closed sets in $X$ are $X, \emptyset, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
10. $g\alpha$ closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.
11. $gsp$ closed sets in $X$ are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
12. $sg$ closed sets in $X$ are $X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
13. $swg$ closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
14. $rg$ closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
15. $gr$ closed sets in $X$ are $X, \emptyset, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
16. $w$ closed sets in $X$ are $X, \emptyset, \{d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}$.
17. $rw$ closed sets in $X$ are $X, \emptyset, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.
18. $rgw$ closed sets in $X$ are $X, \emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

4. On $\beta g^*$ open set

Definition 4.1: A subset $A$ of a topological space $X$ is called $\beta g^*$ open sets if $A$ is $\beta g^*$ closed.

Theorem 4.2: A subset $A$ of a topological space $(X, \tau)$ is $\beta g^*$ open if and only if $B \subseteq gint(A)$ whenever $B$ is $\beta$ closed in $(X, \tau)$ and $B \subseteq A$.

Proof: Let $B \subseteq gint(A)$ and $B \subseteq A$. Then $B \subseteq gint(A)$ implies $gcl(Ac) \subseteq Bc$. Hence $B \subseteq gint(A)$ implies $B \subseteq gcl(Ac)$. Therefore $B \subseteq gint(A)$ implies $B \subseteq gcl(Ac)$.

Theorem 4.3: If $gint(A) \subseteq B \subseteq A$ and $A$ is $\beta g^*$ open subset of $(X, \tau)$ then $B$ is also $\beta g^*$ open subset of $(X, \tau)$.

Proof: Let $B \subseteq gint(A)$ and $B \subseteq A$. Then $B \subseteq gint(A)$ implies $B \subseteq gcl(Ac)$. Hence $B \subseteq gint(A)$ implies $B \subseteq gcl(Ac)$. Therefore $B \subseteq gint(A)$. Thus $B \subseteq gcl(Ac)$.

Theorem 4.4: If a subset $A$ of a topological space $(X, \tau)$ is $\beta g^*$ closed, then $gcl(A)-A$ is $\beta g^*$ open.

Proof: Let $A$ be a $\beta g^*$ closed and let $F$ be $\beta$ open, $gint(A)$ be $\beta g^*$ closed. Since $Fc \subseteq gint(Ac)$ and $gint(Ac) \subseteq Fc$. Then $F \subseteq gint(Ac)$.

Theorem 4.5: If a subset $A$ of a topological space $(X, \tau)$ is $\beta g^*$ closed, then $gcl(A)-A$ is $\beta g^*$ open.

Proof: Let $A$ be a $\beta g^*$ closed and let $F$ be $\beta$ closed such that $F \subseteq gint(A) \cap A$. Then $F \subseteq gcl(A)-A$. Therefore $gcl(A)-A$ is $\beta g^*$ open.
**Theorem 4.6:** If $A$ and $B$ are $\beta g^*$ open sets in $X$ then $A \cap B$ is also $\beta g^*$ open sets in $X$.

**Proof:** Let $A$ and $B$ be two $\beta g^*$ open sets in $X$. Then $A^c$ and $B^c$ are $\beta g^*$ closed sets in $X$. By Theorem 3.12, $A^c \cup B^c$ is a $\beta g^*$ closed in $X$. That is $(A \cap B)^c$ is a $\beta g^*$ closed in $X$. Therefore $(A \cap B)$ is a $\beta g^*$ open set in $X$.

**Theorem 4.7:** If $A \times B$ is a $\beta g^*$ open subset of $(X \times Y, \tau \times \sigma)$, iff $A$ is a $\beta g^*$ open subset in $(X, \tau)$ and $B$ is a $\beta g^*$ open subset in $(Y, \sigma)$.

**Proof:** Let $A \times B$ is a $\beta g^*$ open subset of $(X \times Y, \tau \times \sigma)$. Let $H$ be a closed subset of $(X, \tau)$ and $G$ be a closed subset of $(Y, \sigma)$ such that $H \subseteq A, G \subseteq B$. Then $H \times G$ is closed in $(X \times Y, \tau \times \sigma)$. By assumption $A \times B$ is a $\beta g^*$ open subset of $(X \times Y, \tau \times \sigma)$ and so $H \times G \subseteq \text{gint}(A \times B) \subseteq \text{gint}(A) \times \text{gint}(B)$. That is $H \subseteq \text{gint}(A), G \subseteq \text{gint}(B)$ and hence $A$ is a $\beta g^*$ open subset in $(X, \tau)$ and $B$ is a $\beta g^*$ open subset in $(Y, \sigma)$. Conversely, let $M$ be a closed subset of $(X \times Y, \tau \times \sigma)$ such that $M \subseteq A \times B$. For each $(X, Y) \subseteq M$, $\text{cl}(X) \times \text{cl}(Y) \subseteq \text{cl}(M) = M \subseteq A \times B$. Then the two closed sets $\text{cl}(X)$ and $\text{cl}(Y)$ are contained in $A$ and $B$ respectively. By assumption $\text{cl}(X) \subseteq \text{gint}(A)$ and $\text{cl}(Y) \subseteq \text{gint}(B)$ hold. This implies that for each $(X, Y) \subseteq M$, $(X, Y) \subseteq \text{gint}(A \times B)$. Thus $A \times B$ is a $\beta g^*$ open subset of $(X \times Y, \tau \times \sigma)$.

**References**