(\tau_i, \tau_j) pgprw closed and open sets in bitopological spaces

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Abstract
In this paper, we introduce and investigate the concept of (\tau_i, \tau_j)-pgprw-closed sets which are introduced in a bitopological spaces in analogy with pgprw-closed sets in topological spaces. From now, \tau_p-cl(A) denotes the pre-closure of A of a relative to a topology \tau_p and during this process some of their properties are obtained.

Keywords: pgprw-closed sets, pgprw-open sets, pgprw-closed set, (\tau_i, \tau_j) pgprw-open set

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1. Introduction
The triple (X, \tau_1, \tau_2), where X is a set and \tau_1, \tau_2 are topologies on X is called a bitopological space. Kelly [1] initiated the systematic study of such spaces in 1963. Following the work of kelly on the bitopological spaces, various authors like Arya and Nour [2], Di maio and Noiri [3], Fukutate [4], Nagaveni [5], Maki, Sundaram and Balachandran [6], Shiek john [7], Sampath kumar [8], Patty [9], Arockiarani [10], Gnanambal [11], Reily [12], Rajamani and Viswanthan [13] and Popa [14] have turned their attention to the various concepts of topology by considering bitopological spaces instead of topological spaces.

2. Preliminaries
Let i,j \in \{1,2\} be fixed integers, in a bitopological spaces (X, \tau_1, \tau_2) a subset A of (X, \tau_1, \tau_2) is said to be

(i) (i,j)-g-closed [4] if \tau_j-cl(A) \subseteq U whenever A \subseteq U and U is \tau_i.
(ii) (i,j)-rg-closed [10] if \tau_j-cl(A) \subseteq U whenever A \subseteq U and U is regular open in \tau_i.
(iii) (i,j)gpr-closed [11] if \tau_j-cl(A) \subseteq U whenever A \subseteq U and U is regular open in \tau_i.
(iv) (i,j)wg-closed [5] if \tau_j-cl(A)(\tau_i-int(A)) \subseteq U whenever A \subseteq U and U is \tau_i.
(v) (i,j)W-closed [7] if \tau_j-cl(A) \subseteq U whenever A \subseteq U and U is semi-open in \tau_i.
(vi) (i,j)gp-closed [15] if \tau_j-pcl(A) \subseteq U whenever A \subseteq U and U is \tau_i.
(vii) (i,j)pgprw-closed [16] if \tau_j-pcl(A) \subseteq U whenever A \subseteq U and U is \tau_i.
(vi) Let (X,P,L) be a bitopological space. We say that A \subseteq X is p-semi-open [16] w.r.t L iff there exists a P open set O \subseteq X s.t O \subseteq A \subseteq \tau_i-int(A).
(vi) Similarly A \subseteq X is L-semi open w.r.t P iff there exists L open set O \subseteq X s.t O \subseteq A \subseteq \tau_j-pcl(A).
(vi) Then A is said to be semi-open iff it is both P semi-open w.r.t L and L is semi-open w.r.t P.
(vi) In (X,\tau_1, \tau_2) A \subseteq X is said to be (i,j) pre-open (i,j) [17] p.o iff A \subseteq \tau_i-int (\tau_j-cl(A)),
(ix) i,j=1,2; i\neq j.
(x) (i,j)rgα-closed [18] if \tau_j-rcl(A) \subseteq U whenever A \subseteq U and U is regular α-open in \tau_i.
(xi) Regular open set [19] if A = int(cl(A)) and a regular closed set if A = cl(int(A)).
(xii) Regular generalized α-closed set(briefly,rgα-closed) [20] if \tau_j-cl(A)\subseteq U whenever A \subseteq U and U is regular α-open in X.
(xiii) For a subset A of (X, \tau_1, \tau_2), pgprw-closure of A is denoted by pgprw-cl(A) and defined as pgprw-cl(A)= \cap{G: A\subseteq G, G is pgprw--closed in (X, \tau_1, \tau_2)} or \cap{I: A\subseteq G, G \subseteq pgprw-C(X)}.
3.0 (τ₁, τ₂)pgprw closed sets and their basic properties. 
In this section, we introduce the investigate the concept of (τ₁, τ₂)pgprw-closed sets which are introduced in a bitopological spaces in analogy with pgprw-closed sets in topological spaces. From now on, τ ˓→ pcl(A) denotes the closure of A related to a topology τ.

3.1 Definition: Let i,j ∈ {1,2} be fixed integers, in a bitopological spaces (X,τ₁, τ₂ ), a subset A ⊆ X is said to be (τ₁, τ₂) pgprw-closed set if τ₁-pcl(A) ⊆ G whenever A ⊆ G and G ∈ rga-open (X, τ₂)  
We denote the family of all (i,j) pgprw-closed in a bitopological spaces (X,τ₁, τ₂ ) by Dpgprw (τ₁, τ₂).

3.2 Remark: By setting τ₁=τ₂ in definition 3.1, an (i,j) pgprw-closed set reduces to a pgprw-closed in X. 
First we prove that the class of (i,j) pgprw-closed sets properly lies between the class of (i) pre-closed sets and the class of (i,j) pre-closed sets.

3.3 Theorem: If A is (i,j) pre-closed subset of (X,τ₁, τ₂ ) then A is (i,j) pgprw-closed.

Proof: Let A be (i,j) pre-closed subset of (X,τ₁, τ₂ ).Let G ∈ rga-open (X, τ₂) be s.t A ⊆ G. Since A is pre-closed we have τ₁-pcl(A) = A, which implies τ₁-pcl(A) ⊆ G. Therefore A is (i,j) pgprw-closed. 
The converse of this theorem need not be true as seen from the following example.

3.4 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Then the subset {b} is (1,2) pgprw-closed set but not (1,2) pgprw-closed in the bitopological space (X,τ₁, τ₂ ).

3.5 Theorem: If A is (i,j)- pgprw-closed subset of (X,τ₁, τ₂ ), then A is (i,j) gpr-closed.

Proof: Let A be (i,j) pgprw-closed subset of (X,τ₁, τ₂ ). Let G ∈ regular-open (X, τ₂) be s.t A ⊆ G. Since A is regular-open we have τ₁-pcl(A) ⊆ G and G ∈ rga-open (X, τ₂) then by hypothesis τ₁-pcl(A) ⊆ G, also τ₁-pcl(A) ⊆ τ₁-pcl(A) which implies τ₁-pcl(A) ⊆ G. Therefore A is (i,j) gpr-closed. 
The converse of the theorem need not be true as seen from the following example.

3.6 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Then (a,b), (a,c) are (1,2) is pgprw-closed but not (1,2) pgprw-closed sets in the bitopological spaces.

3.7 Theorem: If A is τ₁-closed subset of a bitopological space (X,τ₁, τ₂ ), then the set A is (i,j) pgprw-closed.

Proof: Let G ∈ regular-open (X, τ₂) be s.t A ⊆ G then by hypothesis τ₁-pcl(A) = A. Which implies τ₁-pcl(A) ⊆ G therefore A is (i,j) pgprw-closed. 
The converse of this theorem need not be true as seen from the follow example.

3.8 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem the subset {b} is (1,2) pgprw-closed set but not (1,2) pgprw-closed in the bitopological space (X,τ₁, τ₂ ).

3.9 Remark: τ₁τ₂ w-closed set and (i,j) pgprw-closed set are independent as seen from the following examples.

3.10 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem the subset {c} is (1,2) pgprw-closed set but not (1,2) pgprw-closed in the bitopological space (X,τ₁, τ₂ ).

3.11 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem the subset {c} is (1,2) pgprw-closed set but not (1,2) pgprw-closed in the bitopological space (X,τ₁, τ₂ ).

3.12 Remark: τ₁τ₂ pgprw-closed set and (i,j) semi-closed set are independent as seen from the following examples.

3.13 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem the subset {b} is semi-closed set but not (1,2) pgprw-closed in (X,τ₁, τ₂ ).

3.14 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem the subset {b} is semi-closed set but not (1,2) pgprw-closed in (X,τ₁, τ₂ ).

3.15 Example: Let X={a,b,c}, τ₁= {X,∅, {a}, {b}, {a,b}} and τ₂ = {X,∅, {a}, {b}, {a,b}}. 
Theorem (a,c) is (1,2) gp-closed set but not (1,2) pgprw-closed in the bitopological space (X,τ₁, τ₂ ).

3.16 Remark: From the above discussion and known results we have the following implication 
A ￬ B means A implies B but not conversely 
A ￬ B means A and B are independent of each other.

3.17 Theorem: If A,B ∈ Dpgprw (i,j) then A∪B ∈ Dpgprw (i,j).

Proof: Let G ∈ rga-open (X, τ₂) be s.t A∪B ⊆ G, then A⊆G and B⊆G since A,B ∈ Dpgprw (i,j). 
We have τ₁-pcl(A) ⊆ G and τ₁-pcl(B) ⊆ G that is τ₁-pcl(A) ∪ τ₁-pcl(B) ⊆ G also 
τ₁-pcl(A) ∪ τ₁-pcl(B) = τ₁-pcl(A∪B) and so τ₁-pcl(A∪B) ⊆ G 
Therefore A∪B ∈ Dpgprw (i,j).

3.18 Remark: The intersection of two (i,j) pgprw-closed sets in generally a (i,j) pgprw-closed set as seen from the following example.

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3.19 Example: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then the subsets $\{a, c\}$ and $\{c\}$ are $(1,2)$ pgprw-closed sets since $\{a, c\} \cap \{c\} = \{c\}$ is a $(1,2)$ pgprw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

3.20 Remark: The family $D_{pgprw}(1,2)$ is generally not equal to the family $D_{pgprw}(2,1)$ seen from the following example.

3.21 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $D_{pgprw}(1,2) = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}\}$ and $D_{pgprw}(2,1) = \{X, \emptyset, \{c\}, \{b\}, \{a, b\}\}$. Therefore $D_{pgprw}(1,2) \neq D_{pgprw}(2,1)$.

3.22 Theorem: If $\tau \subseteq \tau_2$ and $rg\alpha$-open $(X, \tau)$ then $D_{pgprw}(\tau,\tau_2)$.

Proof: Suppose $\{x\}$ is not $rg\alpha$-open $(X, \tau)$ then $\{x\}^c$ is not $rg\alpha$-open $(X, \tau)$.

3.23 Theorem: Let $i,j$ be fixed integers of $\{1,2\}$ for each $x$ of $(X, \tau)$ then $\{x\}^c$ is $rg\alpha$-open $(X, \tau)$ or $\{x\}^c$ is $(i,j)$ pgprw-closed.

Proof: Suppose $\{x\}$ is not $rg\alpha$-open $(X, \tau)$ then $\{x\}^c$ is $rg\alpha$-open $(X, \tau)$.

3.24 Theorem: If $A$ is $(i,j)$ pgprw-closed set then $\tau_j$-$pcl(A)$ and $\tau_i$-$rg\alpha$-open.

Proof: Let $A$ be a $(i,j)$ pgprw-closed set. Suppose $F$ is a non-empty $\tau_i$-$rg\alpha$-open contained in $\tau_j$-$pcl(A)$ then $F$ is $\tau_i$-$rg\alpha$-open. Since $A$ is a $(i,j)$ pgprw-closed set, we have $\tau_j$-$pcl(A) \subseteq F$. Consequently $F \subseteq \tau_j$-$pcl(A)$ which is a contradiction. Hence $\tau_j$-$pcl(A)$ contains no non-empty $\tau_i$-$rg\alpha$-open.

3.25 Example: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{c\}\}$. Then the set $A = \{a, d\}$ then $\tau_j$-$pcl(A) = \{a, d\}$ does not contain any non-empty $\tau_i$-$rg\alpha$-open set and $A$ is not a $(1,2)$ pgprw-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

3.26 Corollary: If $A$ is $(i,j)$ pgprw-closed in $(X, \tau_1, \tau_2)$ then $A$ is $\tau_i$-$rg\alpha$-open set.

Proof: Suppose $A$ is $\tau_i$-$rg\alpha$-open then $\tau_j$-$pcl(A) = A$ and so $\tau_j$-$pcl(A) = \emptyset$ which is a $\tau_i$-$rg\alpha$-open set. Conversely, suppose $\tau_j$-$pcl(A) = A$ is a $\tau_i$-$rg\alpha$-open. Since $A$ is $(i,j)$ pgprw-closed by thm 3.24 $\tau_j$-$pcl(A) = A$ does not contain any non-empty $\tau_i$-$rg\alpha$-open set. Therefore $\tau_j$-$pcl(A) = A = \emptyset$. That is $\tau_j$-$pcl(A) = A$ and hence $A$ is $\tau_i$-$rg\alpha$-open.

3.27 Theorem: In a bitopological space $(X, \tau_1, \tau_2)$, $rg\alpha$-open set $(X, \tau)$ iff every subset of $(X, \tau_1, \tau_2)$ is a $(i,j)$ pgprw-closed.

Proof: Suppose that $rg\alpha$-open set $(X, \tau)\subseteq \{F \subseteq X: F \subseteq \tau_j\}$. Let $A$ be any subset of $X$. Let $rg\alpha$-open $(X,\tau_i)$. b.s.t $A \subseteq G$. Then $\tau_j$-$pcl(G) = G$. Also $\tau_j$-$pcl(A) \subseteq \tau_j$-$pcl(G)$. That is $\tau_j$-$pcl(G) \subseteq G$. Therefore $A$ is a $(i,j)$ pgprw-closed set.

3.28 Theorem: Let $A$ be a $(i,j)$ pgprw-closed subset of a bitopological space $(X, \tau_1, \tau_2)$. If $A$ is $\tau_i$-$rg\alpha$-open, then $A$ is $\tau_i$-$rg\alpha$-open.

Proof: Let $A$ be $rg\alpha$-open $(X, \tau)$. Now, $A \subseteq \tau_i$. Then by hypothesis $\tau_i$-$pcl(A) \subseteq \tau_i$. Therefore $\tau_j$-$pcl(A) = A$. That is $A$ is $\tau_i$-$rg\alpha$-open.

3.29 Theorem: If $A$ is a $(i,j)$ pgprw-closed set and $\tau_i \subseteq \tau_j$-$rg\alpha$-open $(X, \tau, \tau_i)$ then $\tau_j$-$pcl(A)$ and $\tau_i$-$rg\alpha$-closed $(X, \tau, \tau_i)$.

Proof: Let $A$ be $(i,j)$ pgprw-closed and $\tau_i \subseteq \tau_j$-$rg\alpha$-open $(X, \tau, \tau_i)$. Suppose $\tau_j$-$pcl(A)$ then $\tau_i$-$pcl((x)) \cap A = \emptyset$.

3.30 Theorem: If $A$ is $(i,j)$ pgprw-closed set and $A \subseteq \tau_j$-$pcl(A)$, then $B$ is $(i,j)$ pgprw-closed.

Proof: Let $G$ be a $(i,j)$ pgprw-closed set and $\tau_i \subseteq \tau_j$-$rg\alpha$-open $(X, \tau, \tau_i)$. Suppose $\tau_j$-$pcl((x)) \cap A = \emptyset$ for some $x \in \tau_j$-$pcl(A)$. Then $\tau_i$-$pcl((x)) \subseteq \tau_i$-$rg\alpha$-open $(X, \tau, \tau_i)$. Then $\tau_j$-$pcl((x)) \subseteq \tau_j$-$pcl(A)$. Hence $\tau_j$-$pcl(A) \subseteq \tau_j$-$pcl((x))$. This shows that $X \not\in \tau_j$-$pcl(A)$ this contradicts the assumption.

3.31 Theorem: Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $(i,j)$ pgprw-closed in $(X, \tau, \tau_2)$ then $A$ is $(i,j)$ pgprw-closed relative to $Y$ if $Y$ is a $\tau_i$-$rg\alpha$-open.

Proof: Let $A$ be a $(i,j)$ pgprw-closed set and $A \subseteq \tau_j$-$pcl(A)$, then $B$ is $(i,j)$ pgprw-closed.
and Y is $\tau_r$-regular-open set since w.r.t $A \subseteq Y \subseteq X$ where X is a t.s and Y is an open subspace of X.

If $A \subseteq X$ and $\tau_a$ is $\tau_a$-rga-open then $\tau_a$ is $\tau_a$-rga-open. Since A is $(i, j)$ pgprw-closed $\tau_j$-pcl $(A) \subseteq G$. That is $Y \cap \tau_j$-pcl $(A) \subseteq Y \cap G = G$. Also $Y \cap \tau_j$-pcl $(A) = \tau_j$-pcl $(A)$.

Thus $\tau_j$-pcl $(A) \subseteq G$. Hence A is $(i, j)$ pgprw-closed relatively to Y.

3.32 Theorem: In a bitopological space $(X, \tau_1, \tau_2)$ if $\tau_a$-rga-open $(X, \tau_1, \tau_2) = \{X, \emptyset\}$, then every subset of $(X, \tau_1, \tau_2)$ is $(i, j)$ pgprw-closed.

Proof: Let $\tau_a$-rga-open $(X, \tau_1, \tau_2) = \{X, \emptyset\}$, in a bitopological space $(X, \tau_1, \tau_2)$. Let A be any subset of X. To prove that A is an $(i, j)$-pgprw-closed. Suppose $A \neq \emptyset$ then A is $(i, j)$-pgprw-closed.

Suppose $A \neq \emptyset$, then X is the only $\tau_1$-rga-open and $\tau_j$-pcl $(A) \subseteq X$. Hence A is an $(i, j)$ pgprw-closed set.

The converse of the above theorem need not be true in general as seen from the following example.

3.33 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{b\}, \{c\}\}$ then every subset of X is an $(1, 2)$ pgprw-closed set but $\tau_a$-rga-open $(X, \tau_1, \tau_2) = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}$.

3.34 Example: If A is $\tau_1$-open and $(i, j)$ g-closed Then A is $(i, j)$ pgprw-closed.

Proof: Let G be a $\tau_1$-rga-open set s.t $A \subseteq G$, now $A \subseteq A, A$ is $\tau_1$-open and $(i, j)$ g-closed.

We have $\tau_j$-pcl $(A) \subseteq A$, that is, $\tau_j$-pcl $(A) \subseteq G$. Therefore A is $(i, j)$ pgprw-closed.

3.35 Theorem: Suppose $B \subseteq A \subseteq X$. B is a $(i, j)$ pgprw-closed set relative to A and that A is both $\tau_1$-clopen and $\tau_j$-closed. Then B is $(i, j)$-pgprw-closed set in $(X, \tau_1, \tau_2)$.

Proof: Let $\tau_{A, B}$ be the restriction of $\tau_{A, B}$ to A. Let $B \subseteq G$ and G be $\tau_1$-rga-open but it is given that $B \subseteq A \subseteq X$. Therefore $B \subseteq G$ and $B \subseteq A$, which implies $G \subseteq A \nsubseteq G$. Now that we show that $A \nsubseteq G$ is $\tau_1$-open since A is $\tau_1$-open and G is $\tau_1$-open. A $\nsubseteq G$ is $\tau_1$-open since A is $\tau_1$-clopen and G is $\tau_1$-semi-open. A $\nsubseteq G$ is $\tau_1$-semi-open. Thus A is both $\tau_1$-semi-open and A is semi-closed. Hence A $\nsubseteq G$ is $\tau_1$-rga-open set. Since $A \nsubseteq G \subseteq A \subseteq X$. Since w.r.t $A \subseteq Y \subseteq X$, where Y is a topology and X is Y an open subspace of X. If $A \subseteq X$, then A is $\tau_a$-rga-open implies A $\subseteq G$ is $\tau_{A, B}$-rga-open set. Since B is a $(i, j)$ pgprw-closed relative to A

$\tau_{A, B}$-pcl $(B) \subseteq A \nsubseteq G$..

(iii) But $\tau_{A, B}$-pcl $(B) = A \cap \tau_{A, B}$-pcl $(B)$,

Thus $\tau_{A, B}$-pcl $(B) \subseteq G$. Hence B is $(i, j)$ pgprw-closed set in $(X, \tau_1, \tau_2)$.

3.36 Definition: Let $i, j \in \{1, 2\}$ be fixed integers, in a bitopological space $(X, \tau_1, \tau_2)$, a subset $A \subseteq X$, is said to be $(\tau_1, \tau_1)$ pgprw-open if $A^c$ is $(i, j)$ pgprw-closed.

3.37 Theorem: In a bitopological spaces $(X, \tau_1, \tau_2)$ we have the following

(i) every $(i, j)$ pre-open set is $(i, j)$ pgprw-open but not conversely.

(ii) every $\tau_1$-open set is $(i, j)$ pgprw-open but not conversely.

(iii) every $\tau_1$-pgprw open set is $(i, j)$ gpr-open but not conversely.

Proof: The proof follows from the theorems 3.3, 3.5, & 3.7.

3.38 Theorem: If A and B are $(i, j)$ pgprw-open sets, then $A \cap B$ is $(i, j)$ pgprw-open.

Proof: The proof follows from the theorem 3.17.

3.39 Remark: The union of two $(i, j)$ pgprw-open sets is generally an $(i, j)$ pgprw-open set as seen from the following example.

3.40 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{c\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{b\}, \{c\}\}$. Then the subsets $\{b\}$ and $\{a\}$ are $(1, 2)$ pgprw-open sets and $\{b\} \cup \{a\} = \{a, b\}$ is a $(1, 2)$ pgprw-open set in the bitopological space $(X, \tau_1, \tau_2)$.

3.41 Theorem: A subset A of $(X, \tau_1, \tau_2)$ is $(i, j)$ pgprw-open if $F \subseteq \tau_1$-p-int $(A)$, whenever $F$ is $\tau_1$-rga-open set and $F \subseteq A$.

Proof: Suppose that $F \subseteq \tau_1$-p-int $(A)$, whenever $F \subseteq A$ and $F$ is $\tau_1$-rga-open set and to prove that A is $(i, j)$ pgprw-open.

Let G be $\tau_1$-rga-open and $A^c \subseteq G$. Then $G^c \subseteq A$ and $G^c$ is $\tau_1$-rga-open. If $A$ is $\tau_1$-rga-open in $(X, \tau_1)$ then $X-A$ is also $\tau_1$-rga-open, then $G^c \subseteq \tau_1$-p-int $(A)$ that is $(\tau_1$-p-int $(A))^c \subseteq G^c$, since $\tau_1$-pcl $(A^c) = [\tau_1$-p-int $(A)]^c$ thus $A^c$ is $(i, j)$ pgprw-closed that is $A$ is $(i, j)$ pgprw-open.

Conversely, Suppose that A is $(i, j)$-pgprw-open, $F \subseteq A$ and $F$ is $\tau_1$-rga-open. Then $A^c \subseteq F^c$ and $F^c$ is also $\tau_1$-rga-open. If $A$ is $\tau_1$-rga-open in $(X, \tau_1)$, then $X-A$ is also $\tau_1$-rga-open.

Since $A$ is $(i, j)$-pgprw-closed, we have $(\tau_1$-p-cl $(A^c)) \subseteq F^c$ and so $F \subseteq \tau_1$-p-int $(A)$, since $(\tau_1$-p-cl $(A^c)) = (\tau_1$-p-int $(A))^c$.

3.42 Theorem: Let A and G be two subsets of a bitopological spaces $(X, \tau_1, \tau_2)$. If the set A is $(i, j)$ pgprw-open, then $G \subseteq X$, whenever G is $\tau_1$-rga-open and $(\tau_1$-p-int $(A)) \cup A^c \subseteq G$.

Proof: Let A be $(i, j)$ pgprw-open, G be the $\tau_1$-rga-open and $(\tau_1$-p-int $(A)) \cup A^c \subseteq G$. Then $G^c \subseteq (\tau_1$-p-int $(A)) \cup A^c$ - $A^c$. Since $A^c$ is $(i, j)$ pgprw-closed and $G^c$ is $\tau_1$-rga-open. w.r.t if A is $(i, j)$ pgprw-closed, then $\tau_1$-pcl $(A)$- A contains no non-empty $\tau_1$-rga-open set. It follows that $G^c = \emptyset$, therefore $G = X$.

The converse of the theorem need not be true as seen from the following example.

3.43 Example: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\tau_1$-rga-open $(X, \tau_1, \tau_2) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.
If $A = \{a,b\}$ then the only $\tau_1$ rg$\alpha$-open set containing $\tau_2 = p$-int$(A) \cup A'$ is $X$. But $A$ is not (1,2) pgprw-open set in $(X, \tau_1, \tau_2)$.

3.44 Theorem: If a subset $A$ of $(X, \tau_1, \tau_2)$ is (i,j) pgprw-closed, then $\tau_1$-p-cl$(A)$ - $A$ is (i,j) pgprw-open.

Proof: Let $A$ be a (i,j) pgprw-closed subset in $(X, \tau_1, \tau_2)$. Let $F$ be a $\tau_1$ rg$\alpha$-open set such that $F \subseteq \tau_1$-p-cl$(A)$ - $A$. Since w.k.t if $A$ is (i,j) pgprw-closed then $\tau_1$-p-cl$(A)$ - $A$ contains non-empty $\tau_1$ rg$\alpha$-open set then $\tau_1$-p-cl$(A)$ - $A$ is (i,j) pgprw-open then $F = \emptyset$.

Therefore $F \subseteq \tau_1$-p-int$(\tau_1$-p-cl$(A)$ - $A)$ and by theorem 3.41: $\tau_1$-p-cl$(A)$ - $A$ is (i,j) pgprw-open.

The converse of the above theorem need not be true as seen from the following example.

3.45 Example: Let $X = \{a,b,c,d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}\}$ and $\tau_2 = \{Y, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$ for the subset $A = \{a,b\}$ in $(X, \tau_1, \tau_2)$ $\tau_1$-p-cl$(A)$ - $A = X - \{a,b\} = \{c\}$ is (1,2) pgprw-open but $A = \{a,b\}$ is not (1,2) pgprw-closed.

3.46 Theorem: If $\tau_1$-p-int$(A) \subseteq \tau_2$ and $A$ is (i,j) pgprw-open in $(X, \tau_1, \tau_2)$ then $B$ is (i,j) pgprw-open.

Proof: Let $F$ be $\tau_1$ rg$\alpha$-open s.t $F \subseteq B$. Now $F \subseteq A$ that is $F \subseteq A$. Since $F$ is (i,j) pgprw-open by theorem 3.41 $F \subseteq \tau_1$-p-int$(A)$ by hypothesis $\tau_1$-p-int$(A) \subseteq B$.

Therefore $\tau_1$-p-int$(\tau_1$-p-int$(A) \subseteq \tau_1$-p-int$(B)$. That is $\tau_1$-p-int$(A) \subseteq \tau_1$-p-int$(B)$ and hence $F \subseteq \tau_1$-p-int$(A)$ again by thm 3.41; $B$ is (i,j) pgprw-open set in $(X, \tau_1, \tau_2)$.

3.47 Corollary: Let $A$ and $B$ be subsets of a space $(X, \tau_1, \tau_2)$ if $B$ is (i,j) pgprw-open and $\tau_1$-p-int$(B) \subseteq A$, then $A \cap B$ is (i,j)-pgprw-open.

Proof: Let $B$ be (i,j) pgprw-open and $\tau_1$-p-int$(B) \subseteq A$ that is $\tau_1$-p-int$(B) \subseteq A$ then $\tau_1$-p-int$(B) \subseteq A \cap B$.

Also $\tau_1$-p-int$(B) \subseteq A \cap B$ and $B$ is (i,j) pgprw-open by thm 3.46 then $A \cap B$ is (i,j)-pgprw-open.

3.48 Theorem: Every singleton point set in a space $(X, \tau_1, \tau_2)$ is either (i,j) pgprw-open.

Proof: Let $(X, \tau_1, \tau_2)$ be a bitopological space. Let $x \in X$ to prove $\{x\}$ is either (i,j) pgprw-open that is to prove $X - \{x\}$ is either (i,j) pgprw-closed which follow from the statement i.e, $i,j$ be fixed integers of (1,2) for each $x$ of $(X, \tau_1, \tau_2)$, $\{x\}$ is a rg$\alpha$-open in $(X, \tau)$.

4.0 $(\tau_1, \tau_2)$ pgprw-closure in bitopological spaces

4.1 Definition: Let $(X, \tau_1, \tau_2)$ be a bitopological space and $i,j \in \{1,2\}$ be fixed integers. For each subset $E$ of $X$ define $(\tau_1, \tau_2)$-pgprw cl$(E) = \cap \{A: E \subseteq D_{pgprw}(i,j)-pgprw-cl(E)\}$.

4.2 Theorem: If $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ then

(i) $(i,j)$ pgprw-cl$(X - X)$ and $(i,j)$-pgprw-cl$(\emptyset)$.

(ii) $A \subseteq (i,j)$-pgprw-cl$(A)$ and $(i,j)$ if $B$ is any $(i,j)$ pgprw-closed set containing $A$ then $(i,j)$-pgprw-cl$(A) \subseteq B$.

Proof: Follows from the definition 4.1.

4.3 Theorem: Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ and $i,j \in \{1,2\}$ be fixed integers. If $E \subseteq B$, then $(i,j)$-pgprw-cl$(A) \subseteq (i,j)$-pgprw-cl$(A)$.

Proof: Let $E \subseteq B$, by definition 4.1 $(i,j)$-pgprw-cl$(B) = \cap \{F: E \subseteq D_{pgprw}(i,j)\}$.

If $E \subseteq D_{pgprw}(i,j)$ since $E \subseteq B$, $A \subseteq F \subseteq D_{pgprw}(i,j)$, we have $(i,j)$ pgprw-cl$(A) \subseteq F$.

That is $(i,j)$-pgprw-cl$(A) \subseteq (i,j)$ pgprw-cl$(B)$.

4.4 Theorem: Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $\tau_2 \cap (X, \tau_1, \tau_2)$ and $\tau_1$ be subsets of (X, $\tau_1$, $\tau_2$) and $\tau_1, \tau_2$ and $\tau_2$-open (X, $\tau_1$, $\tau_2$) and $\tau_2$-open (X, $\tau_1$, $\tau_2$) then (1,2) pgprw-cl$(A) \subseteq (2,1)$ pgprw-cl$(A)$.

Proof: By definition 4.1 $(1,2)$-pgprw-cl$(A) = \cap \{F: A \subseteq D_{pgprw}(1,2)\}$. Since $\tau_1 \subseteq \tau_2$. Since w.k.t, if $\tau_1 \subseteq \tau_2$ and $\tau_1$-rg$\alpha$-open in $(X, \tau_1, \tau_2)$ then $D_{pgprw}(\tau_1, \tau_2) \subseteq D_{pgprw}(\tau_2, \tau_1)$ and $A \subseteq F$ implies $A \subseteq D_{pgprw}(2,1)$ Therefore

$\cap \{F: A \subseteq D_{pgprw}(1,2)\} \subseteq \cap \{F: A \subseteq D_{pgprw}(2,1)\}$, that is $(1,2)$ pgprw-cl$(A) \subseteq (2,1)$ pgprw-cl$(A)$.

4.5 Theorem: Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i,j \in \{1,2\}$ be fixed integers, then $A \subseteq (i,j)$ pgprw-cl$(A) \subseteq (i,j)$ pgprw-cl$(A)$.

Proof: By definition 4.1, it follows that $A \subseteq (i,j)$ pgprw-cl$(A)$.

Now to prove that $(i,j)$ pgprw-cl$(A) \subseteq (i,j)$ pgprw-cl$(A)$ by definition of closure, $(i,j)$ pgprw-cl$(A) = \{F \subseteq X, A \subseteq F; F$ is $\tau_2$-closed$\}$ if $A \subseteq F$ and $F$ is $\tau_1$-closed then $F$ is (i,j)-pgprw-closed, as every $\tau_1$-closed set is (i,j) pgprw-closed.

Therefore $(i,j)$ pgprw-cl$(A) \subseteq \cap \{F \subseteq X, A \subseteq F; F$ is $\tau_1$-closed$\}$. Then $\tau_1$-p-cl$(A)$ hence $(i,j)$ pgprw-cl$(A) \subseteq (i,j)$ pgprw-cl$(A)$.

4.6 Remark: Containment relation in the above theorem may be proper as seen from the followin example.

4.7 Example: Let $X = \{a,b,c,d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ then $\tau_2$-closed sets are $\{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ and (1,2) pgprw-closed sets are $\{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$.

Take $A = \{a,b\}$ then $\tau_2$-cl$(A) = X$.

And (1,2) pgprw-cl$(A) = \{b,c,d\}$. Now $A \subseteq (1,2)$ pgprw-cl$(A)$ but $A \neq (1,2)$ pgprw-cl$(A)$.

Also (1,2) pgprw-cl$(A) \subseteq \tau_2$ - cl$(A)$ but (i,j) pgprw-cl$(A) \neq \tau_1$ - cl$(A)$.

4.8 Theorem: Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i,j \in \{1,2\}$ be fixed integers. If $A$ is (i,j)pgrpw-closed then $(i,j)$ pgprw-cl$(A) = A$.

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Proof: Let A be a (i,j) pgprw-closed subset of (X, τ_1, τ_2) w.r.t A \subseteq (i,j)-pgprw-cl(A) also A \subseteq A and A is (i,j) pgprw-closed by theorem 4.2(iii) (i,j) pgprw-cl(A) \subseteq A.
Hence (i,j) pgprw-cl(A) = A.

4.9 Remark: The converse of the above theorem 4.8 need not be true as seen from the following example

4.10 Example: Let X = \{a, b, c\}, τ_1 = \{ X, \emptyset, \{a\}, \{b\}, \{a, b\}\} τ_2 = \{ X, \emptyset, \{a\}\} then (1,2) pgprw-closed sets {b, c} Take A = \{a\} Now (1,2)pgprw-cl(A) = X \cap \{a, c\} \cap \{a\} = \{a\}
But \{a\} is not a (1,2) pgprw-closed set.

4.11 Theorem: The operator (i,j)-pgprw-closure in definition 4.1(i) is the kurulowski closure operator on X.

Proof:
(i) Let (i,j) pgprw-cl(\emptyset) = \emptyset by theorem 4.2(i)
(ii) E \subseteq (i,j)pgprw-cl(E) for any subset E of X by theorem 4.2(ii)
(iii) Suppose E and F are two subsets of (X, τ_1, τ_2) then it follows from theorem 4.3 that
(i,j) pgprw-cl(E) \subseteq (i,j)pgprw-cl(EU F) and that (i,j) pgprw-cl(F) = (i,j)-pgprw-cl(EU F) hence we have (i,j)-pgprw-cl(E) \cup (i,j)-pgprw-cl(F) \subseteq (i,j)-pgprw-cl(F).
Now if x \in (i,j)-pgprw-cl(E) \cup (i,j)-pgprw-cl(F) then x \in (i,j)-pgprw-cl(E), it follows that there exist A, B \subseteq \text{pgprw}(i,j) s.t E \subseteq A, x \in A and F \subseteq B, x \notin B. Hence EU \subseteq AUB, x \notin AUB. Since AUB is (i,j)-pgprw-closed since w.r.t A and B, then AUBe \subseteq (i,j)-pgprw-cl(EUF) then we have (i,j)-pgprw-cl(EU F) \subseteq (i,j)-pgprw-cl(E) \cup (i,j)-pgprw-cl(F).

From the above discussions we have (i,j)-pgprw-cl(EU F) = (i,j)-pgprw-cl(E) \cup (i,j)-pgprw-cl(F).
(iv) Let E be any subset of (X, τ_1, τ_2) by the definition of (i,j)-pgprw-closure, (i,j)-pgprw-cl(E) = \{ A \subseteq X : E \subseteq A \ \text{pgprw}(i,j) \}. If E \subseteq A \ \text{pgprw}(i,j) \} then (i,j)-pgprw-cl(E) \subseteq A. Since A is a (i,j)-pgprw-closed set containing (i,j)-pgprw-cl(E) by theorem 4.2(iii) (i,j)-pgprw-cl(E) \subseteq A. Hence (i,j)-pgprw-cl(E) \subseteq \{ A \subseteq X : E \subseteq A \ \text{pgprw}(i,j) \}. Conversely, (i,j)-pgprw-cl(E) \subseteq \{ A \subseteq X : E \subseteq A \ \text{pgprw}(i,j) \} is true by theorem 4.2(iii), then we have (i,j)-pgprw-cl(E) = (i,j)-pgprw-cl(E). Hence (i,j)-pgprw-cl(E) is a kuraowski closure operator on X.

From the this theorem (i,j)pgprw-closure defines the new topology on X.

4.12 Definition: Let 1 \leq j \leq 2 be two fixed integers. Let τ_{pgprw}(i,j) be topology on X generated by (i,j) pgprw-closure in the usual manner. That is τ_{pgprw}(i,j) = \{ E \subseteq X : (i,j)-pgprw-cl(E) = E \}.

4.13 Theorem: Let (X, τ_1, τ_2) be a bitopological space and i, j \in \{1, 2\} be two fixed integers, then τ_{pgprw}(i,j) = τ_{pgprw}(i,j).

Proof: Let Ge τ_{pgprw}(i,j). It follows that G^c is τ_{pgprw}(i,j) pgprw-closed. Therefore (i,j)-pgprw-cl(G^c) = G^c by theorem 4.8 that is Ge τ_{pgprw}(i,j) and hence τ_{pgprw}(i,j) = τ_{pgprw}(i,j).

4.14 Remark: Containment relation in the above theorem 4.13 may be proper as seen from the following example.

4.15 Example: Let X = \{a, b, c\}, τ_1 = \{ X, \emptyset, \{a\}, \{b\}, \{a, b\}\} τ_2 = \{ X, \emptyset, \{a\}\} then (1,2) pgprw-closed sets are \{a\} \subseteq \text{pgprw}(1,2) but \{b\} \notin \text{pgprw}(1,2).

4.16 Theorem: Let (X, τ_1, τ_2) be a bitopological space and i,j \in \{1, 2\} be two fixed integers. If a subset E of X is (i,j) pgprw-closed then E is τ_{pgprw}(i,j)-open.

Proof: Let a subset E of X be (i,j) pgprw-closed by theorem 4.8 (i,j) pgprw-cl(E) = E that is (i,j)-pgprw-cl(E) = E \subseteq E. It follows that E \subseteq τ_{pgprw}(1,2). Therefore E is τ_{pgprw}(i,j)-closed.

4.17 Theorem: For any point x of (X, τ_1, τ_2),\{x\} is τ_1 - \text{rg} \alpha open or τ_{pgprw}(i,j)-open.

Proof: Let x be any point of (X, τ_1, τ_2), since w.r.t i,j be fixed integers of \{1, 2\} for each x of (X, τ_1, τ_2),\{x\} is \text{rg} \alpha open in (X, τ_1) or \text{rg} \alpha open in (X, τ_2) then \{x\} is τ_1 - \text{rg} \alpha open that is \{x\} is τ_{pgprw}(i,j)-closed, by above thm 4.16 Therefore \{x\} is τ_1 - \text{rg} \alpha open or τ_{pgprw}(i,j)-open.

4.18 Theorem: If τ_1 \subseteq τ_2 and \text{rg} \alpha open(X, τ_1) \subseteq \text{rg} \alpha open(X, τ_2) in(X, τ_1, τ_2) then τ_{pgprw}(2,1) \subseteq τ_{pgprw}(1,2).

Proof: Let Ge τ_{pgprw}(2,1), then (2,1)-pgprw-cl(G^c) = G^c to prove that G \subseteq τ_{pgprw}(1,2) that is to prove (1,2)-pgprw-cl(G^c) = G^c. Now (1,2)-pgprw-cl(G^c) = \{ F \subseteq G^c : F \subseteq \text{pgprw}(1,2) \} since τ_1 \subseteq τ_2 and \text{rg} \alpha open(X, \tau_1) \subseteq \text{rg} \alpha open(X, \tau_2) by thm 3.22 D_{pgprw}(2,1) \subseteq D_{pgprw}(1,2).

Thus \{ F \subseteq G^c : F \subseteq \text{pgprw}(1,2) \} \subseteq \{ F \subseteq G^c : F \subseteq \text{pgprw}(2,1) \} that is (1,2)-pgprw-cl(G^c) \subseteq (2,1) Ppgprw-cl(G^c) and so, (1,2) pgprw-cl(G^c) = G^c. Conversely, G^c \subseteq (1,2)-pgprw-cl(G^c) is true by the theorem 4.2(ii) then we have (1,2)-pgprw-cl(G^c) = G^c that is Ge τ_{pgprw}(2,1) and hence τ_{pgprw}(2,1) \subseteq τ_{pgprw}(1,2).

References
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