On ternary quadratic equation \( x^2 + xy + y^2 = 12z^2 \)

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Abstract
The Ternary quadratic Diophantine equation given by \( x^2 + xy + y^2 = 12z^2 \) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary, Quadratic, integral solutions, Polygonal Numbers.

1. Introduction
The ternary quadratic Diophantine equation offers an unlimited field for research because of their variety \([1, 3]\). For an extensive review of various problems, one may refer \([1, 20]\). This communication concerns with yet another interesting ternary quadratic equation \( x^2 + xy + y^2 = 12z^2 \) for determining its infinitely many non-zero distinct integral solutions. Also a few interesting relations among the solutions have been exhibited.

2. Notations used:
- \( t_{3,n} \)-Polygonal number of rank \( n \) with size \( m \)
- \( P_{3n} \)-Tetrahedral number of rank \( n \)
- \( P_{4n} \)-Square pyramidal number of rank \( n \)
- \( P_{5n} \)-Pentagonal pyramidal number of rank \( n \)

3. Method of Analysis
The Ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solution is
\[
\begin{align*}
x^2 + xy + y^2 &= 12z^2 \\
\end{align*}
\]

Pattern – I
On substitution of linear transformations \( u \neq v \neq 0 \)
\[
\begin{align*}
x &= u+3v, & y &= u-3v \\
\end{align*}
\]
In (1) leads to \( u^2 + 3v^2 = 4z^2 \) (2)

The corresponding solutions of (3) is the form
\[
\begin{align*}
u &= a^2 - 6ab - 3b^2 \\
v &= a^2 + 2ab - 3b^2 \\
z &= a^2 + 3b^2 \\
\end{align*}
\]
(4)

In view of (4), the solution of (1) can be written as
\[
\begin{align*}
x &= 4a^2 - 12b^2 \\
y &= 6b^2 - 12ab - 2a^2 \\
z &= a^2 + 3b^2 \\
\end{align*}
\]
(5)

Instead of (2) using the transformations \( x = u-3v, \ y = u+3v \) in (1), we get again (3) only. Thus, the integer solutions of (1) are obtained as
\[
\begin{align*}
x &= -2a^2 - 12ab + 6b^2 \\
y &= 4a^2 - 12b^2 \\
z &= a^2 + 3b^2 \\
\end{align*}
\]
(6)
A few interesting properties observed are as follows:
I) \( 5x(2a, a) + y(2a, a) \equiv 0 \pmod{4} \)
II) \( x(a, 1) - y(a, 1) - 12t_{1, a} \equiv 0 \pmod{6} \)
III) \( y(1, B) + 2(1, B) - t(64, B) + t(34, B) \equiv 0 \pmod{15} \)
IV) \( x(A, 2) - t_{320t}, A + t_{198}, A \equiv -24 \pmod{26} \)

Each of the following expression represents a nasty numbers.
\[ 4z(a, b) - x(a, b) \]
\[ 3\{x(a, b) + 4z(a, b)\} \]
\[ x(a, a) + 2y(a, a) \]
\[ y(a, a) - 4z(a, a) \]

**Pattern – II**
Equation (3) can be written as
\[ 3v^2 - 3z^2 = z^2 - u^2 \]
\[ 3(v + z)(v - z) = (z + u)z - u \]

Four different choices of solution obtained are as follows:

**Choice I**
\[ X = 2A^2 - 6B^2 + 12AB \]
\[ Y = 12B^2 - 4A^2 \]
\[ Z = A^2 + 3B^2 \]

A few interesting properties observed are as follows.
I) \( x(1, B) + 12t_{3, B} \equiv 2 \pmod{18} \)
II) \( x(5, B) + 12t_{3, B} \equiv 16 \pmod{66} \)
III) \( x(10, B) + 12t_{3, B} \equiv 74 \pmod{126} \)
IV) \( x(A, 2) - 4t_{3, A} \equiv -2 \pmod{22} \)
V) \( x(A, 4) - 4t_{3, A} \equiv -4 \pmod{46} \)
VI) \( 2x(A, A+1) + y(A, A+1) - 48t_{3, A} \equiv 0 \)
VII) \( 2x(A, A+1) + y(A, A+1) - 48t_{3, A} \equiv 0 \)
VIII) \( 2x(A, A+1)(A+2) + y(A, A+1)(A+2) - 144P_5 \equiv 0 \)
IX) Each of the following expression represents a nasty numbers
(a) \( 2x(A, B) + y(A, B) \)
(b) \( y(A, B) + 4z(A, B) \)
(c) \( y(3B, B) \)

**Choice II**
\[ x = -4A^2 + 12B^2 \]
\[ y = 2A^2 - 6B^2 - 12AB \]
\[ z = A^2 + 3B^2 \]

**Choice III**
\[ X = 2A^2 - 6B^2 + 12AB \]
\[ Y = 12B^2 - 4A^2 \]
\[ Z = -A^2 - 3B^2 \]

**Choice IV**
\[ X = 2A^2 - 6B^2 - 12AB \]
\[ Y = -4A^2 + 12B^2 \]
\[ z = -3B^2 - A^2 \]

**Pattern – III**
Equation (3) can be written as
\[ u^2 + 3v^2 = 4z^2 \pmod{1} \]
Assume that \( z = a^2 - 3b^2 \)
Write 1 as \( 1 = ((1 + i\sqrt{3})(1 - i\sqrt{3})) \)
\[ u + i\sqrt{3}v = \frac{(1 + i\sqrt{3})(1 + i\sqrt{3})(a + i\sqrt{3}b)^2}{2(10)} \]
Equating the real and imaginary parts in
\[ u = -a^2 + 3b^2 - 6ab \]
\[ v = a^2 - 3b^2 - 2ab \]
Substituting (11) and (12) in (2), the corresponding integer solution of (1) are given by
\[ x = 2a^2 - 6b^2 - 12ab \]
\[ y = -4a^2 + 12b^2 \]
\[ z = a^2 + 3b^2 \]

A few interesting properties observed are as follows.
I) \( x(1, B) + 12t_{3, B} \equiv 2 \pmod{6} \)
II) \( x(2, B) + 12t_{3, B} \equiv 8 \pmod{18} \)
III) \( x(3, B) + 12t_{3, B} \equiv 18 \pmod{30} \)
IV) \( x(4, B) + 12t_{3, B} \equiv 22 \pmod{42} \)
V) \( x(a, 1) - 4t_{3, a} \equiv -6 \pmod{14} \)
VI) \( x(a, 2) - 4t_{3, a} \equiv -24 \pmod{26} \)
VII) \( x(a, 3) - 4t_{3, a} \equiv -16 \pmod{38} \)
VIII) \( x(a, 4) - 4t_{3, a} \equiv -46 \pmod{50} \)
IX) Each of the following expression represents a nasty numbers
(a) \( x(a, a) - 2y(a, a) \)
(b) \( y(a, b) + 4z(a, b) \)
(c) \( 2x(a, b) + y(a, b) \)
(d) \( x(a, a) - 2z(a, a) \)

**Pattern – IV**
Again, Equation (3) can be written as
\[ u^2 + 3v^2 = 4z^2 \pmod{1} \]
Assume that \( z = a^2 + 3b^2 \)
Write 1 as \( 1 = ((1 + i\sqrt{3})(1 - i\sqrt{3})(a + i\sqrt{3}b)^2 \)
\[ \frac{49}{7} \]
Equating the real and imaginary parts in
\[ u = 1/7(33b^2 - 11a^2 - 30ab) \]
\[ v = 1/7(5a^2 - 15b^2 - 22ab) \]

Our interest is to obtain the integer solutions, so that the values of \( u \) and \( v \) are integers for suitable choices of the parameters \( a \) and \( b \).

Put \( a = 7, b = 7 \)
\( u = 231B^2 - 77A^2 - 210AB \)
\( v = 35A^2 - 154AB - 105B^2 \)
\( z = 49A^2 + 174B^2 \)
Substituting (20) and (21) in (2), the corresponding integer solutions of (1) are given by
\[ x = 28A^2 - 84B^2 - 672AB \]
\[ y = 546B^2 - 182A^2 - 252AB \]
\[ z = 49A^2 + 147B^2 \]

The equation (23) represents non-zero distinct integral solution of (1) on two parameters.
A few interesting properties observed are as follows.
I) \( x(A, 1) - 56t_{3, a} \equiv -84 \pmod{700} \)
II) \( x(A, 3) - 56t_{3, a} \equiv -756 \pmod{2044} \)
Pattern – V

Equation (3) may be equivalent to

\[ u^4 + 3v^2 = (2z)^2 \]  

(24)

Which is satisfied by

\[ u = 3p^2, \quad q^2 \]  

(25)

\[ v = 2pq \]  

(26)

\[ z = 1/2 \left(3p^2 + q^2\right) \]  

(27)

Our interest is to obtain the integer solutions, so that the values of \( z \) are integers for suitable choices of the parameters \( p \) and \( q \).

put \( p = 2A, \quad q = 2B \) in (25), (26) and (27) we get

\[ u = 12A^2 - 4B^2 \]  

(28)

\[ v = 8AB \]  

(29)

\[ z = 6A^2 + 2B^2 \]  

(30)

Substituting (28) and (29) in (2), the corresponding integer solutions of (1) are given by

\[ x = 12A^2 - 4B^2 + 24AB \]  

\[ y = 12A^2 - 4B^2 - 24AB \]  

\[ z = 6A^2 + 2B^2 \]  

A few interesting properties observed are as follows.

(i) \( x = (A,A) - 24A^2, \quad y = 0 \) (mod 84)

(ii) \( x = (A,7) - 24A^2, \quad y = 0 \) (mod 156)

(iii) \( y = 1(B) + 8tA^2, \quad z = 0 \) (mod 20)

(iv) \( y = 3(B) + 8tA, \quad z = 0 \) (mod 68)

(v) \( x = (A,B) - y = (A, AB), \quad z = 0 \) (mod 12)

(vi) \( x = (A,A+1) - y = (A, A(A+1) - 96A, \quad z = 0 \) (mod 12)

(vii) \( x = (A,(A+1)A), \quad y = (A, A(A+1)(A+2)) - 288PA^2, \quad z = 0 \) (mod 12)

4 Conclusion

In this paper we have presented five different patterns of non-zero distinct integer solutions of the ternary quadratic equation given by \( x^2 + xy + y^2 = 12z^2 \) To conclude, one may search for other patterns of solutions and their corresponding properties.

5 Reference