International Journal of Applied Research 2016; 2(7): 933-938

International Journal of Applied Research

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2016; 2(7): 933-938
www.allresearchjournal.com
Received: 13-05-2016
Accepted: 14-06-2016

Sneha S Kadiya
Research Scholar at Calorx University, Ahmedabad, Gujarat, India.

Dharmin R Patel
Research Scholar at Calorx University, Ahmedabad, Gujarat, India.

Dr. Pradeep J Jha
Prof. of Emeritus at Calorx University, Ahmedabad, Gujarat, India.

Correspondence
Dr. Pradeep J. Jha
Prof. of Emeritus at Calorx University, Ahmedabad, Gujarat, India.

Matrix operations on generator matrices of known sequences and important derivations

Sneha S Kadiya, Dharmin R Patel and Dr. Pradeep J Jha

Abstract
Contents of this paper target on introducing some new, based on certain geometrical results, infinite integer sequences. In addition to this, some distinct algebraic properties related to these sequences and also to some known sequences are also explicitly mentioned. It further works on searching for algebraic properties and eigen values of generator matrices of different exponential order. The utmost centre of associated mathematical work is found in alignment of (1) successive terms of different sequences and their recurrence relations (2) generator matrices of different exponents and association of their eigen values with one and/or two different given sequences, and (3) number of useful mathematical results inter- relating these sequences.

Keywords: Jha sequence, Pell sequence, Fibonacci sequence, NSW sequence *, CDS Sequence**, Sequence of hypotenuse of Fermat Triangle, Generator matrix, and Recurrence relation
(* Newman Shank and William Sequence, ** A special sequence identified by students of Dr. Jha)

Introduction

The contents of this paper have been classified in two sections, denoted as (A) and (B). In the first part we introduce some new sequences, which originate from known geometrical results, and their fundamental mathematical properties like general term, generator matrices along with their algebraic properties, and their corresponding eigen values in our terminology as mentioned above. We continue establishing some interesting mathematical properties too. In the second part (B) we carry out mathematical operations on the generator matrices and obtain symmetrical results which have been mentioned in the tabular form using structural properties of these known sequences, already introduced in part (A), and enlist some important inter-relations of these sequences.

Notations: For the mathematical proceedings of this paper, we have used the following notations.
Area = Δ, s = (a +b +c)/2 = Semi-perimeter, in-radius = r, t(n)= t_n = n^{th} term

1 (A): Introduction to some Known Sequences

1.1 Jha Sequence: Jha sequence is an infinite sequence of positive integers which is observed and extracted from on working with Fermat family of right triangles. It is a sequence that has been found very useful in coordination of some geometrical properties of right triangles and algebraic relation of some known sequence like Pell sequence, and NSW sequences.
Some of the initial members of the right triangles of Fermat family are as follows and one can easily find the origin of Jha sequence from generating their in-radius.

Table 1

S. No.	Sides of the triangle			Area = Δ	s = (a +b+c)/2	in-radius = r
	a	b	h	(a. b)/2		
[*1	0	1	1	0	1	0]
2	3	4	5	6	6	1
3	20	21	29	210	35	6
4	119	120	169	7140	204	35

[* Being very useful in further work in finding the ratio of successive terms of 'Jha' sequence as n →∞, which is a convergent sequence tending to a constant (irrational number), to establish parity with the successive terms of some known sequences and also useful for application purpose justifying sin 0 = 0, we have considered a = 0]

It is important to note that
(1) The difference between the consecutive legs; i.e. |a-b| = 1
(2) The corresponding terms of the in-radius forms an infinite sequence known as 'Jha' sequence.
General term of Jha sequence, denoted as j (n) or j_n, is given by the following relation.

$$J(n) = j_n = \frac{\sqrt{2}}{8}\left[\left(3+2\sqrt{2}\right)^{n-1} - \left(3-2\sqrt{2}\right)^{n-1}\right] \text{ for } n \in N \qquad (1)$$

Some terms of the Jha sequence are 0, 1, 6, 35, 204,………. with j(1) = 0, j(2) = 1, j(3) = 6 etc.

The recurrence relation inherent in the terms is $j_{n+2} = 6\, j_{n+1} - j_n,\ n \in N$ (2)

The generator matrix of the sequence is denoted as J is found from recurrence relation as follow

$$\begin{pmatrix} j_3 \\ j_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} j_2 \\ j_1 \end{pmatrix};\ J = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \qquad (3)$$

Here we enlist certain mathematical properties of the matrix J.
p1. $|J| = \left|\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}\right| = \mathbf{1}$ and Eigen Values are $E_{\mathbf{j1}} = 3 \pm 2\sqrt{2}$
p2. $J^2 = \begin{pmatrix} 35 & -6 \\ 6 & -1 \end{pmatrix}$; $|j^2| = 1$ and Eigen Values are $E_{\mathbf{j2}} = 17 \pm 12\sqrt{2}$ (4)
p3. $J^3 = \begin{pmatrix} 204 & -35 \\ 35 & -6 \end{pmatrix}$; $|j^3| = 1$ and Eigen Values are $E_{\mathbf{j2}} = 99 \pm 70\sqrt{2}$
p4. The most general form of the generator matrix is given by $J^n = \begin{pmatrix} j_{n+2} & -j_{n+1} \\ j_{n+1} & -j_n \end{pmatrix}$; $|j^n| = 1$ (5)**
Looking to the pattern of the eigen values we shall discuss its pattern after introducing Pell sequence and the sequence 3, 17, 99, ……(CDS sequence)

1.2 Pell Sequence: Much is said about and discussed on is the Pell sequence which results as the rational approximation sequence converging to $\sqrt{2}$ and is obtained from a particular form of Pell equation $x^2 - 2y^2 = \mp 1$. The ratio (x/y) provides a close approximation to $\sqrt{2}$; where x and y are natural numbers. The sequence of approximations is shown as follows.
1, 3/2, 7/5, 17/12, 41/29……… In this form the denominator of each fraction is a Pell number. To keep pace and maintain uniformity we introduce ' 0 'as the first term and write terms of Pell sequence as 0, 1, 2, 5, 12, 29, 70,…….
The general term of the Pell sequence $P_n = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^{n-1} - \left(1-\sqrt{2}\right)^{n-1}\right]$ for $n \in N$ (6)
The inherent recurrence relation in the terms of the sequence is $P_{n+2} = 2\,P_{n+1} + P_n$ for $n \in N$ **(7)** with $P_1 = 0$, $P_2 = 1$

The generator matrix of the sequence is denoted as P is found from recurrence relation as follows.

$$\begin{pmatrix} P_3 \\ P_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} P_2 \\ P_1 \end{pmatrix};\ P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad (8)$$

Here we enlist certain mathematical properties.
p1. $P = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$; $|P| = -1$ and Eigen Values are $\mathbf{E_{P1}} = 1 \pm 1\sqrt{2} = (P_1 + P_2) \pm P_2\sqrt{2}$
p2. $P^2 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$; $|P^2| = 1$ and Eigen Values are $E_{\mathbf{P2}} = 3 \pm 2\sqrt{2} = (P_2 + P_3) \pm P_3\sqrt{2}$
p3. $P^3 = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix}$; $|P^3| = -1$ and Eigen Values are $E_{\mathbf{P3}} = 7 \pm 5\sqrt{2} = (P_3 + P_4) \pm P_4\sqrt{2}$

The general form of the generator matrix is given by
$P^n = \begin{pmatrix} P_{n+2} & P_{n+1} \\ P_{n+1} & P_n \end{pmatrix}$; with $|P^n| = (-1)^n$ for all $n \in N$

Using the symmetry in the set of eigen values, we write a very important relation connecting the eigen values of matrices of different exponents of the basic generator matrix.
$E_{Pn} = (P_n + P_{n+1}) \pm P_{n+1}\sqrt{2}$ for all $n \in N$; where P_n, for each n, is the term of Pell sequence.

1.3 CDS Sequence: This sequence is an outcome of our extended work on Pythagorean triplets. It is a sequence of such positive integers when each of its individual term, if considered, as a shorter leg in a right triangle the next leg is always a perfect square.
General term of CDS sequence denoted as CDS (n) or CDS_n is given by the following relation.

$$CDS(n) = CDS_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^{n} + \left(3-2\sqrt{2}\right)^{n}\right] \text{ for } n \in N \qquad (9)$$

Some terms of the CDS sequence are 3, 17, 99, 577, ………. With CDS (1) = 3, CDS (2) = 17, CDS (3) = 99 etc.

[**For any primitive Pythagorean triplet (a, b, h), if a is a member of CDS sequence then b is a perfect square of an integer that satisfies $a^2 + b^2 = h^2$.] As stated above, considering each term of CDS sequence as the shorter leg of a right triangle, the next leg b ($= x^2$ for some $x \in N$) of the same triangle is given by the terms of the sequence 4, 144, 4900, …….. for which each term is a perfect square. This important and unique property empowers the introduction of CDS sequence**]**

The recurrence relation inherent in the terms is $C_{n+2} = 6\,C_{n+1} - C_n$ for $n \in N$ (10)

[Note: At this stage it is important to note that the recurrence relation of Jha sequence and that of for CDS sequence is the same.]

The generator matrix of the sequence is denoted as **C** is found from recurrence relation as follows $\begin{pmatrix} C_3 \\ C_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_2 \\ C_1 \end{pmatrix}$; with the generator matrix C = $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$

Here we enlist certain mathematical properties.

p1. $|C| = |\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}| = 1$ and Eigen Values are $E_{c1} = 3 \pm 2\sqrt{2}$

p2. $C^2 = \begin{pmatrix} 35 & -6 \\ 6 & -1 \end{pmatrix}$; $|C^2| = 1$ and Eigen Values are $E_{c2} = 17 \pm 12\sqrt{2}$

p3. $C^3 = \begin{pmatrix} 204 & -35 \\ 35 & -6 \end{pmatrix}$; $|C^3| = 1$ and Eigen Values are $E_{c3} = 99 \pm 70\sqrt{2}$

p4. $C^4 = \begin{pmatrix} 1189 & -204 \\ 204 & -35 \end{pmatrix}$; $|C^3| = 1$ and Eigen Values are $E_{c4} = 577 \pm 408\sqrt{2}$

p5. The most general form of the generator matrix is given by $C^n = \begin{pmatrix} C_{n+2} & -C_{n+1} \\ C_{n+1} & -C_n \end{pmatrix}$; $|C^n| = 1$ and its Eigen values of C^n are given by $E_{cn} = (\text{CDS } n \pm P_{2n+1}.\ \sqrt{2})$

1.4 Fibonacci Sequence: The most known and widely application oriented sequence is Fibonacci sequence. Some initial terms of Fibonacci sequence are 0,1,1,2,3,5,8,13, ……….

The general term denoted as f_n is given by the following result.

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] \text{ for } n \in N \quad (11)$$

The recurrence relation in the terms of the sequence is $f_n = f_{n-1} + f_{n-2}$ for $n > 2$ and $f_1=0$, $f_2=1$

The generator matrix, denoted as 'F', of Fibonacci sequence is F= $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ satisfying the continuation of terms of the sequence $\begin{pmatrix} f_3 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} f_2 \\ f_1 \end{pmatrix}$

Here we enlist certain mathematical properties.

p1. $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; $|P| = -1$ and Eigen Values are $\mathbf{E_{F1}} = \left(\frac{1\pm\sqrt{5}}{2}\right) = \left(\frac{(f_3+f_1)\pm(f_2)\sqrt{5}}{2}\right)$

p2. $F^{\mathbf{2}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $|P^2| = 1$ and Eigen Values are $E_{F2} = \left(\frac{3\pm\sqrt{5}}{2}\right) = \left(\frac{(f_4+f_2)\pm(f_3)\sqrt{5}}{2}\right)$

p3. $F^{\mathbf{3}} = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$; $|P^3| = 1$ and Eigen Values are $E_{F3} = \left(\frac{4\pm2\sqrt{5}}{2}\right) = \left(\frac{(f_5+f_1)\pm(f_4)\sqrt{5}}{2}\right)$

p4. The most general form of the generator matrix is given by $F^{\mathbf{n}} = \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}$

$|F^n| = 1$and its Eigen Values are $E_{Fn} = \left(\frac{(fn+2+fn)\pm(fn+1)\sqrt{5}}{2}\right)$ for all $n \in N$

1.5 Fermat Hypotenuse sequence

This sequence is an output of the work on finding the general term of the hypotenuses of right triangles of Fermat family.*

As per the discussion above the hypotenuses of right triangles of Fermat family are

1, 5, 29, 169, 985…..

We denote the general term by h (n) or h_n.

We have some important observations here.

(1) These terms are even ordered terms of Pell- sequence.

(2) If J_{n+1} and J_n are two consecutive terms of Jha sequence then $|J_{n+1} - J_n| = h_n$

(3)It has a recurrence relation observed successively from the third term.

$h_1 = 1$, $h_2 = 5$ and $h_{n+2} = 6\,h_{n+1} - h_n$ for $n \in N$ (12)

[Note: Recurrence relation of Fermat Hypotenuse sequence is the same as that of Jha sequence and also that of CDS sequence.]

The most general term of the hypotenuse sequence is given by the following formula.

$h_n = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)(3+2\sqrt{2})^{n-1} + (\sqrt{2}-1)(3-2\sqrt{2})^{n-1}\right]$ (13)

The generator matrix of the sequence is denoted as **H** is found from recurrence relation as follows.

H = $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$; we have $h_1 = 1$, $h_2 = 5$, $h_3 = 29$
[*A Fermat family of right triangles is the one whose non- hypotenuse sides differ by unity. If **a** and **b** are two consecutive sides increasing order of magnitude then |b – a|= 1]
We have the successive terms as follows.

$\begin{pmatrix} h_3 \\ h_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_2 \\ h_1 \end{pmatrix}$ and so on.

Here we enlist certain mathematical properties.

p1. H = $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ and |H| = 1 and Eigen Values are $E_{h1} = 3 \pm 2\sqrt{2}$
p2. $H^2 = \begin{pmatrix} 35 & -6 \\ 6 & -1 \end{pmatrix}$; $|H^2| = 1$ and Eigen Values are $E_{h2} = 17 \pm 12\sqrt{2}$ (14)
p3. $H^3 = \begin{pmatrix} 204 & -35 \\ 35 & -6 \end{pmatrix}$; $|H^3| = 1$ and Eigen Values are $E_{h3} = 99 \pm 70\sqrt{2}$

p4. It is clearly observed that the entries of the matrix H and its exponents are nothing but the terms of Jha sequence and using the fact we write general form of the generator matrix is

$H^n = \begin{pmatrix} j_{n+2} & -j_{n+1} \\ j_{n+1} & -j_n \end{pmatrix}$; where j_n is the n^{th} term of Jha sequence.
In the same pattern we have $|H^n| = 1$ (15)
Looking to the pattern of the eigen values one can easily identify that the Eigen values of each matrix in the above set-up are the algebraic sum of **CDS** sequence and odd terms of Pell sequence (for n > 1). This is the most important observation.

1.6 NSW * Sequence
[*Newman Shank William sequence]
This sequence is the outcome of combined work of the above mentioned mathematicians.
Some initial terms of NSW sequence are 1, 7, 41, 239, 1393, 8119, 47321,……
For the reference work to this paper we denote the terms by the symbol N(n) or N_n for $n \in N$
$N_1 = 1$, $N_2 = 7$, $N_3 = 41$…..
It has been found very useful in interpreting and comparing
(1)Recurrence relation of Jha sequence, Fermat sequence of hypotenuse, and CDS sequence etc
(2) The terms of the sequence independently shows the sum of two successive legs of Fermat right triangles
(3) a sequence resulting as the sum of first (2n+1) terms is square 0f $(n+1)^{th}$ term
$N_1 + N_2 + N_3 = 1 + 7 + 41 = N_2^{\,2} = (7)^2 = 49$ and $\sum_{i=1}^{i=5} N_i = (N_3)^2 = (41)^2 = 1681$
This property is very useful and interesting which is rarely found.
Some important features of this sequence are as follows.

p1: It has the same recurrence relation in its terms as is found in that of Jha sequence, CDS sequence and Fermat sequence of hypotenuse]
For n>2, $N_n = 6N_{n-1} - N_{n-2}$
p2: As can be derived from recurrence relation and also as an application of Jha sequence, general term of the NSW sequence, denoted as N_n is

$$N_n = \frac{1}{2}\left[(1+\sqrt{2})(3+2\sqrt{2})^{n-1} + (1-\sqrt{2})(3-2\sqrt{2})^{n-1}\right] \quad (16)$$

p3: The generator matrix of the sequence is N = $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$; Where $N_1 = 1$, $N_2 = 7$, $N_3 = 41$…..
We have the successive terms as follows.
$\begin{pmatrix} n_3 \\ n_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$ and so on.

Here we enlist certain mathematical properties.

1. $N = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$ and $|N| = 1$ and Eigen Values are $E_{n1} = 3 \pm 2\sqrt{2}$

2. $N^2 = \begin{pmatrix} 35 & -6 \\ 6 & -1 \end{pmatrix}$; $|N^2| = 1$ and Eigen Values are $E_{n2} = 17 \pm 12\sqrt{2}$ (17)

3. $N^3 = \begin{pmatrix} 204 & -35 \\ 35 & -6 \end{pmatrix}$; $|N^3| = 1$ and Eigen Values are $E_{n2} = 99 \pm 70\sqrt{2}$

p4: It is clearly observed that the entries of the matrix N and its exponents are nothing but the terms of Jha sequence and using the fact we write general form of the generator matrix is

$N^n = \begin{pmatrix} J_{n+2} & -J_{n+1} \\ J_{n+1} & -J_n \end{pmatrix}$; where j_n is the n^{th} term of Jha sequence.

In the same pattern we have $|N^n| = 1$ (18)

[Looking to the pattern of the eigen values one can easily identify that the Eigen values of each matrix in the above set-up are the algebraic sum of CDS sequence and odd terms of Pell sequence (for n > 1). This is the most important observation.]

1.7 Comments on Eigen Values

During the discussion of all the different sequences from 1.1 'Jha' Sequence up to 1.6 'NSW' sequence; we have observed that the sequences which have the same generator matrix have the same eigen values and it is obvious. The important point drawing sharp attention is the eigen values of n^{th} exponent of the generator matrix. The first part of the eigen value is the corresponding term of 'CDS' sequence and the second part is either $2\sqrt{2}$ times corresponding terms of 'Jha' sequence or it is the $(2n+1)^{th}$ term of the 'Pell' sequence for $n \geq 1$ If G stands for the generator matrix then eigen values of G^n are $E(G^n) = CDS(n) \pm 2\sqrt{2}\,(j(n)) = CDS(n) \pm \sqrt{2}.\,(P(2n+1))$

2 (B): Inter-Relationship between the sequences

Now we study inter relationship between these different sequences and identify important characteristics.

We have discussed, in order, (1) Jha sequence, (2) Pell sequence, (3) CDS sequence, (4) Fibonacci sequence, (5) Fermat sequence of Hypotenuse, and (6) NSW sequence

Some important derivations connecting the terms of the above mentioned sequences are listed below. All these results are algebraically derived using elementary properties.

S. No.	Name of the Sequence	General Term	Recurrence Relation	Some initial terms
1	Pell Seq.	$P_n = \frac{\sqrt{2}}{4}\left[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right]$	$P_{n+2} = 2P_{n+1} + P_n$ $n \in N$	0, 1, 2, 5, 12, 29, 70, 169,...
2	Jha Seq.	$J_n = \frac{\sqrt{2}}{8}\left[(3+2\sqrt{2})^{n-1} - (3-2\sqrt{2})^{n-1}\right]$	$j_{n+2} = 6j_{n+1} - j_n$, $n \in N$	0, 1, 6, 35, 204, 1189, 6930…..
3	NSW Seq.	$N_n = \frac{1}{2}\left[(1+\sqrt{2})(3+2\sqrt{2})^{n-1} + (1-\sqrt{2})(3-2\sqrt{2})^{n-1}\right]$	$N_{n+2} = 6N_{n+1} - N_n$ $n \in N$	1, 7, 41, 239, 1393, 8119…….
4	CDS Seq.	$CDS_n = \frac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right]$	$C_{n+2} = 6C_{n+1} - C_n$ $n \in N$	3, 17, 99, 577, 3363,…..
5	Hypotenuse	$H_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})(3+2\sqrt{2})^{n-1} + (1-\sqrt{2})(3-2\sqrt{2})^{n-1}\right]$	$H_{n+2} = 6H_{n+1} - H_n$ $n \in N$	1, 5, 29, 169, 985, 5741…….

(1) Different sequences regulated by the same recurrence relation have the same generator matrix.

E.G. Jha sequence, CDS sequence, Fermat hypotenuse sequence, and NSW sequence –all these have the same recurrence relation.

$T_{n+2} = 6\,T_{n+1} - T_n$ for all $n \in N$

As a result their generator matrix is $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$

Remaining properties have been derived using algebraic coordination between general terms of these sequences. All these results can be verified using general terms of the above sequences.

(2) $P_n + P_{n+1} = N_n$

(3) $P_{n+1} + P_{n+2} = CDS_n$

(4) $P_{2n} = H_n$

(5) $\frac{1}{2}P_{2n-1} = J_n$

(6) $J_n + J_{n+1} = N_n$

(7) $J_{n+1} - J_n = H_n$
(8) $\sum_{k=1}^{n-1} H_k = J_n$
(9) $\sum_{k=1}^{n-1}(-1)^{n+1} N_{k+1} = J_n$
(10)$(CDS_n)^2 + (P_{2n+1})^4 = (P_{2n}P_{2n+2})^2$
(11) $N_n + N_{n+1} = 4(P_{2n+1}) = 8(J_n)$
(12) Jha sequence is very useful in finding sum of finite number of terms of Pell sequence.

Conclusion

In this note what we have broadly observed and established that some sequences are closely inter-related and probably our efforts are converging to find one sequence that dominates all sequences and can operate independently incorporating basic inherent properties of all member sequences. We have also identified applications of some sequences such that the sequence of their reciprocal closely aligns with properties of Cauchy sequence. Broadly saying that this interesting topic, as we keep on working, is highly open-ended and there are many fair scopes for further work. In the galaxy of sequences space there are different independent, non-overlapping and non-intersecting sequences 'stand alone' families which participate only for certain common mathematical properties.

References

1. Hazara AK, Matrix Algebra. Calculus and Generalized Inverse (Part-I), Viva books, 2009.
2. Hazara AK, Matrix Algebra. Calculus and Generalized Inverse (Part-II), Viva books, 2009.
3. Patel DR, Kadiya SS, Patel CB, Jha PJ. Member Triangles of Fermat Family and Geometric Inter-relationship with standard Sequences, Proceeding of the International Conference on-Emerging Trends in scientific Research, ISBN., 978-2-642-24819-9, December, 2015.
4. Achesariya V, Jha PJ. Three super Operator Matrices and their properties and Applications, Proceeding of the International Conference on-Emerging Trends in scientific Research, ISBN. 978-2-642-24819-9. 2015.
5. Shah SH, Prajapati DP, Achesariya VA, Dr. Jha PJ. Classification of matrices on the basis of special Characteristics, International Journal of Mathematics Trends and Technology. 2015; 19:91-101.
6. Ranna A, Trivedi, Shailesh A, Bhanotar, Dr. Pradeep J Jha. Pythagorean Triplets-Views, Analysis, and Classification, IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728, ISSN: 2319-765X, 2015; 11(2):III54-63.