A mathematical model on the two phase coronary blood flow through arterioles during angina

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Abstract
In the present paper we have formulated the coronary blood flow in heart. Through arterioles the coronary circulation consists of the blood vessels that supply blood to remove blood from the heart muscle. The blood enters coronary arterioles from coronary artery. Upadhyay V. and Pandey P.N. have considered that the blood flow as two phase, one of red blood cells and other is plasma. They have also applied the Herschel Bulkley Non-Newtonian model in bio fluid mechanical set-up. We have collected a clinical data case of angina for hematocrit v/s blood pressure. The graphical presentation for particular parametric value is much closed to the clinical observation. The overall presentation is in tensorial form and solution technique adopted is analytical as well as numerical. The role of hematocrit is explicit the determination of blood pressure in the case of coronary heart disease with special reference to angina.

Keywords: Coronary blood flow, arterioles Herschel Bulkley, hematocrit, angina, non – Newtonian model, circulatory system

Introduction
The human heart is a muscular organ containing four chambers that is situated just to the left of the midline of the thoracic cavity. The upper two chambers atria are divided by a wall like structure called the intertribal septum. The lower two chambers verticals are divided by a similar structure called the inter ventricular septum. Between each atrium and ventricle, valves allow blood to flow in one direction, preventing backflow. The wall of the heart has three layers –The Epicardium, myocardium, endocardium [1]. Blood low oxygen and high in carbon dioxide enters the right side of the heart and is pumped into the pulmonary circulation. After oxygenation into the lungs and some removal of carbon dioxide, it returns to the left side of the heart. The left ventricle pumps blood out of the heart to the rest of body.
Blood is a complex fluid consisting of particulate solids suspended in a non-Newtonian fluid. The particulate solids are red blood cells (RBCs), white blood cells (WBCs) and platelets. The fluid is plasma, which itself is a complex mixture of proteins and other intergradient in an aqueous base 50% of plasma and 45% of the blood cells and 45% of the blood is RBCs and there is a few parts of the other cells. Which are ignorable, So one phase of the blood is plasma and second phase of the blood is RBCs. Two phase coronary blood flow is study of measuring the blood pressure if hemoglobin known. The percentage of volume covered by blood cells in the whole blood is called hematocrit.

The coronary circulation consists of the blood vessels that supply blood to and remove blood from the heart muscle itself. Although blood fills the chambers of the heart or myocardium is so thin that it requires coronary blood vessels to deliver blood deep into the myocardium. The vessels that supply blood high in oxygen to the myocardium are known as coronary arteries [2]. These arteries when healthy are capable of auto regulation to maintain coronary blood flow at levels appropriate to the needs of the heart muscle.

Other blood vessels include arteries, veins, venules and capillaries. The structure of an arteriole is: Arterioles are tiny branches of arteries that lead to capillaries. Arterioles are under the control of the sympathetic nervous system, and constrict and dilate to regulate blood flow. Arterioles, as a group, are the most highly regulated blood vessels in the body and contribute the most to overall blood pressure. Arterioles respond to a wide variety of chemical and electrical messages and are constantly changing size to speed up or slow down blood flow. Arterioles are an integral part of the circulatory system, which is a closed system in the sense that blood does not enter or leave the system during its journey from the heart, to the body, and back again. In such a system, a continuous flow of the same liquid can be pumped through the loop again and again.

![Image of Arterioles]

**Fig 2: structure of Arterioles**

Angina is chest pain or discomfort that occurs if an area of your heart muscle doesn’t get oxygen rich blood, generally due to obstruction or spasm of coronary arteries [3]. The main cause of angina pectoris is coronary artery disease. Due to atherosclerosis of the arteries feeding the heart. In some cases angina can be extremely serious and has been known to cause death. People that suffer from average to severe cases of angina have an increased percentage of death before the age of 55, usually around 60%. There are three types of angina:-

1. **Stable Angina**: Also known as effort angina this refers to the more common understanding of angina related to myocardial ischemia. Typical presentation of stable angina is that of chest discomfort associated with some activity [running, walking etc.] with minimal or non-existent symptoms at rest or with administration of sublingual nitroglycerin [4].

2. **Unstable Angina**: Unstable angina (UA) is defined as angina pectoris that changes or worsens [5]. It has at least of these three features-

   - It occurs at rest (or with minimal exertion ), usually lasting >10 min;
   - It is severe and new on set [i.e.; within the prior 4-6 weeks];
   - It occurs with a crescendo pattern.

3. **Micro vascular Angina**: Micro vascular angina or Angina syndrome X is characterized by angina like chest pain but it appears to be the result of spasm in the tiny blood vessels of the heart, arms and legs [6].

Equation of motion for two phase blood flow
The hydro dynamical pressure \( p \) between two the phase of the blood can be supposed to be uniform because the both phases i.e. blood cells and plasma are always in equilibrium state in blood(14), Taking viscosity coefficient of blood cells to be \( \eta_c \) and applying the equation of motion for two phase of blood cells as follows-

\[
X \rho_c \frac{\partial \mathbf{u}}{\partial t} + (X \rho_c \mathbf{V}_i) \cdot \nabla \mathbf{u} = -Xp \mathbf{g}_i + X \eta_c (\mathbf{g}_i \mathbf{V}_i) \]

\[
\rho_m \frac{\partial \mathbf{u}_m}{\partial t} + (\rho_m \mathbf{V}_i) \cdot \nabla \mathbf{u}_m = -\rho_m \mathbf{g}_i + \eta_m (\mathbf{g}_i \mathbf{V}_i) \quad \ldots (6)
\]

Taken the viscosity coefficient of plasma to be \( \eta_p \). The equation of motion for plasma will be as follows-

\[
(1-X) \rho_p \frac{\partial \mathbf{u}_p}{\partial t} + (1-X) \rho_p \mathbf{V}_j \cdot \nabla \mathbf{u}_p = -(1-X)p \mathbf{g}_j + (1-X) \eta_p (\mathbf{g}_j \mathbf{V}_j) \quad \ldots (7)
\]

Now adding equation (6) and (7) and using relation (4), the equation of the motion for blood flow with the both phases will be as follow-

\[
\rho_m \frac{\partial \mathbf{u}_m}{\partial t} + (\rho_m \mathbf{V}_i) \cdot \nabla \mathbf{u}_m = -\rho_m \mathbf{g}_i + \eta_m (\mathbf{g}_i \mathbf{V}_i) \quad \ldots (8)
\]

where \( \eta_m \) is viscosity of blood cells, \( \eta_p \) viscosity of plasma and \( \eta_{pm} = X\eta_c + (1-X)\eta_p \) is the viscosity coefficient of blood as a mixture of two phases. As the viscosity of blood flow decreases, the velocity of blood increases. The velocity of blood decreases successively because of the fact that arterioles. These vessels are relative far enough from the heart. Hence the pumping of the heart on these vessels is relativity low. Secondly these vessels relativity narrow down more rapidly in this situation, the blood cells line up on the axis to build up rouleaux. Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten time. because of the fact that arterioles. These vessels are relative far enough from the heart. Hence the pumping of the heart on these vessels is relativity low. Secondly these vessels relativity narrow down more rapidly in this situation, the blood cells line up on the axis to build up rouleaux. Hence a yield stress is produced. Though this yield stress is very small, even then the viscosity of blood is increased nearly ten times.

Mathematical Modeling
As the velocity of Blood flow decreases, the viscosity of blood increases. The velocity of blood decreases successively. The Herschel Bulkley law holds good on the two, phase blood flow though arterioles. Whose constitutive equation is as follows-

\[
T = \eta_m e^\varepsilon + T_p (T \leq T_p) \quad \text{and} \quad e = 0 \quad (T > T_p)
\]

where \( T_p \) is the yield stress.

When strain rate \( e = 0(T < T_p) \)
A core region is formed which flows just like a plug. Let the radius of the plug be \( r_p \). The stress acting on the surface of plug will be \( T_p \). Equating the forces acting on the plug, we get

\[
P r_p^2 = T_p 2 \pi r_p
\]

\[
r_p = 2 \frac{r_p}{r_p} \quad \ldots (9)
\]
The constitutive equation for test part of the blood vessel is

\[ T = \eta_0 \varepsilon + T_p \]

or

\[ T - T_p = \eta_0 \varepsilon = T_e \]

where, \( T_e \) = effective stress

whose generalized form will be as follows

\[ T_{ij} = -\rho g_{ij} + T_e{ij} \]

where, \( T_e{ij} = \eta_0 (e_{ij})^n \)

While \( e_{ij} = g_{ik} V_{ki} \)

Where the symbols have their usually meanings.

Now we describe the basic equations for Herschel Bulkely blood flow as follows:-

Equation of Continuity-

\[ \frac{1}{\sqrt{g}} \sqrt{(g V_i)}_{,i} = 0 \quad \ldots \text{(10)} \]

Equation of Motion-

\[ \rho_m \frac{\partial V_i}{\partial t} + \eta_m V_j V_{ij} = -T_e{ij} \]

Where all the symbols have their usual meaning.

**Solution and Discussion**

Since, the blood vessels are cylindrical. The above governing equations have to be transformed into cylindrical co-ordinates. As we know earlier:-

\[ x^1 = r, \ x^2 = \Theta, \ x^3 = z \]

Matrix of metric tensor in cylindrical co-ordinates is as follows:-

\[
\begin{bmatrix}
  g_{ij}
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & r^2 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

While matrix of conjugate matrix tensor is as follow:

\[
\begin{bmatrix}
  g_{ij}
\end{bmatrix} = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \frac{1}{r^2} & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

Whereas the christoffel’s symbol of 2nd kind are as follow:-

\[
\iota_{22} = \iota_r, \ \{ 22 \} = \{ 22 \} = \frac{1}{r}
\]

Remaining others are zero.

The equation of continuity

\[ \frac{\partial \rho}{\partial z} = 0 \]

The equation of motion

r-component 0 = \[ \frac{\partial \rho}{\partial z} \]

z component 0 = \[ \frac{\partial \rho}{\partial z} + \frac{\eta_0}{\rho} \left( \frac{\partial \rho}{\partial z} \right)^n \]

Here, this fact has been taken in view that the blood flow is axially Symmetric in arteries concerned, i.e.

\( V_\Theta = 0 \) and \( V_r, V_z \) and \( p \) do not depend upon \( \Theta \).

We get \( V_z = v(r) \) and \( p = p (z) \) and

\[ 0 = \frac{\partial \rho}{\partial z} + \frac{\eta_0}{\rho} \left[ \frac{\partial v}{\partial z} \right]^n \]

Since, pressure gradient \[ \frac{\partial \rho}{\partial z} = p \]

\[ r \left( \frac{\partial v}{\partial r} \right)^n = -pr^2/2\eta_0 + A \] we apply

Boundary condition: at \( r = 0 \), \( v = v_0 \) then \( A = 0 \)

\[ v = \frac{\partial \rho}{\partial r} \]

Replace \( r \) from \( r = r - r_p \)

\[ v = \left( \frac{\partial r_p v_p}{\partial r} \right)^n \]

Integratin g above666 equation (12) under the no slip boundary condition- \( v = 0 \) at \( r = R \) so as to get:

\[ V = \left( \frac{\partial r_p v_p}{\partial r} \right)^n \]

This is the formula for velocity of blood flow in arterioles.

Putting \( r = r_p \) to get the velocity \( V_p \) of plug flow as follows:-

\[ V_p = \left( \frac{\partial r_p v_p}{\partial r} \right)^n \left( R - r_p \right)^{\frac{n+1}{2}} \]

This is the formula for velocity of blood flow in coronary arteries.

\[ Q = \int_{r_p}^{R} 2\pi r v_r dr + \int_{r_p}^{R} 2\pi r v_r dr \]

**Result (Bio-physical interpretation)**

**Observation-** Hematocrit vs. blood pressure from an authorized Jabalpur hospital & Research centre By Dr. Umesh Agrawal Patient Name-Mr. Krishna kumar (Age-52 years old)

**Diagnosis- Coronary Artery Disease**

<table>
<thead>
<tr>
<th>Date</th>
<th>HB%</th>
<th>B.P</th>
<th>3 HB</th>
<th>Hematocrit</th>
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<tbody>
<tr>
<td>22/05/2016</td>
<td>17.6</td>
<td>130/80</td>
<td>3*17.6</td>
<td>52.8</td>
</tr>
<tr>
<td>25/05/2016</td>
<td>17.2</td>
<td>125/80</td>
<td>3*17.6</td>
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</tr>
<tr>
<td>27/05/2016</td>
<td>16.5</td>
<td>115/80</td>
<td>3*16.5</td>
<td>49.5</td>
</tr>
<tr>
<td>30/05/2016</td>
<td>16.2</td>
<td>110/80</td>
<td>3*16.2</td>
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</tr>
<tr>
<td>02/06/2016</td>
<td>15.2</td>
<td>120/90</td>
<td>3*15.2</td>
<td>45.6</td>
</tr>
</tbody>
</table>

The flow flux phase blood flow in coronary arteries.

\[ Q = \int_{r_p}^{R} 2\pi r v_r dr + \int_{r_p}^{R} 2\pi r v_r dr \]
\[ Q = 425\text{ml/min} = 1, \ r_p = 1/3 \]

\[ \eta_0 = 0.0015 \text{ pascal-sec) } \]

\[ \eta_m = 0.035 \text{ (pascal-sec)} \]

\[ H = 49.6, P = 115 \]

\[ \eta_m = \eta_c X + \eta_p (1-X) \text{ Where, } X = H/100, \ X = 49.6/100 = 0.496 \]

\[ 0.104587 = (268.3113)^{1/n} \]

we get \( n = 0.0112 \)

\[ Q = 425\text{ml/min} \]

Gustafson, Daniel R. (1980)

\[ \eta_0 = 0.0015 \text{ pascal-sec) } \]

Glenn Elert (2010)

\[ \eta_m = 0.035 \text{ (pascal-sec)} \]

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<th>51.6</th>
<th>49.5</th>
<th>48.6</th>
<th>45.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>BP drop</td>
<td>25</td>
<td>22.5</td>
<td>17.5</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>

\[ Q = 425\text{ml/min} = 1, \ r_p = 1/3 \]

Conclusion
A simple survey of the graph between blood pressure drop and hematocrit in cardiac patient shows that when hematocrit is increased the blood pressure drop is also increased.

Acknowledgement
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Reference
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