On Soft SGB-Closed sets in soft topological spaces

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Abstract
This paper focuses on soft sgb-closed sets and soft sgb-open sets in soft topological spaces and to investigate its properties. Further soft sgb-T$_{1/2}$ space is introduced and its basic properties are discussed.

Keywords: Soft closed, soft generalized closed, soft sgb-closed, soft sgb-T$_{1/2}$ space, soft topological spaces.

1. Introduction

In this paper, we define a new class of sets called soft sgb-closed sets and study the relationships with other soft sets. Also we introduce soft sgb-T$_{1/2}$ and study its basic properties.

2. Preliminaries
Let U be an initial universe set and E be a collection of all possible parameters with respect to U, where parameters are the characteristics or properties of objects in U. Let P(U) denote the power set of U, and let $A \subseteq E$.

Definition 2.1. [11] A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For a particular $e \in A$, F (e) may be considered the set of e-approximate elements of the soft set (F, A).

Definition 2.2: [3] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if
(i) $A \subseteq B$ and
(ii) $\forall e \in A, F(e) \subseteq G(e)$.

We write (F, A) $\subseteq$ (G, B), (F, A) is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A) and is denoted by $(F, A) \supseteq (G, B)$.

Definition 2.3: [10] For two soft sets (F, A) and (G, B) over a common universe U, union of two soft sets of (F, A) and (G, B) is the soft set (H, C), where $C = A \cup B$ and $\forall e \in C$,
\[ H(e) = \begin{cases} 
F(e) \text{ if } e \in A - B \\
G(e) \text{ if } e \in B - A \\
F(e) \cup G(e) \text{ if } e \in A \cap B 
\end{cases} \]

We write \((F, A) \cup (G, B) = (H, C)\).

**Definition 2.4:** [3] The intersection \((H, C)\) of two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\) denoted by \((F, A) \cap (G, B)\) is defined as \(C = A \cap B\) and \(H(e) = F(e) \cap G(e)\) for all \(e \in C\).

**Definition 2.5:** [14] Let \(\tau\) be the collection of soft sets over \(X\) then \(\tau\) is called a soft topology on \(X\) if \(\tau\) satisfies the following axioms:
1. \(\phi, X\) belong to \(\tau\)
2. The union of any number of soft sets in \(\tau\) belongs to \(\tau\).
3. The intersection of any two soft sets in \(\tau\) belongs to \(\tau\).
   The triplet \((X, \tau, E)\) is called a soft topological space over \(X\).
   For simplicity, throughout the work we denote the soft topological space \((X, \tau, E)\) as \(X\).

**Definition 2.6:** [14] Let \((X, \tau, E)\) be soft space over \(X\). A soft set \((F, E)\) over \(X\) is said to be soft closed in \(X\), if its relative complement \((F, E)'\) belongs to \(\tau\). The relative complement is a mapping \(F: E \rightarrow P(X)\) defined by \(F(e) = X - F(e)\) for all \(e \in A\).

**Definition 2.7:** [6] Let \(X\) be an initial universe set, \(E\) be the set of all parameters. Let \((F, A)\) and \((G, B)\) be soft sets over the common universe set \(U\) and \(A, B\) be the set of all parameters.

**Definition 2.8:** [6] Let \((X, \tau, E)\) be a soft topological space over \(X\) and the soft interior of \((F, E)\) denoted by \(\text{Int}(F, E)\) is the union of all soft open subsets of \((F, E)\). Clearly, \((F, E)\) is the largest soft open set over \(X\) which is contained in \((F, E)\).

**Definition 2.9:** [6] Let \(U\) be the common universe set and \(E\) be the set of all parameters. Let \((F, A)\) and \((G, B)\) be soft sets over the common universe set \(U\) and \(A, B \subseteq E\). Then \((F, A)\) is a subset of \((G, B)\), denoted by \((F, A) \subseteq (G, B)\). \((F, A)\) equals \((G, B)\), denoted by \((F, A) = (G, B)\) if \((F, A) \subseteq (G, B)\) and \((G, B) \subseteq (F, A)\).

**Definition 2.10:** A soft subset \((A, E)\) of \(X\) is called
(i) a soft generalized closed (soft g-closed) \([8]\) if \(\text{Cl}(A, E) \subseteq (U, E)\) whenever \((A, E) \subseteq (U, E)\) is soft open in \(X\)
(ii) a soft semi open \([2]\) if \((A, E) \subseteq \text{Cl}(\text{Int}(A, E))\).
(iii) a soft regular open \([5]\) if \((A, E) = \text{Int}(\text{Cl}(A, E))\).
(iv) a soft a-open \([5]\) if \((A, E) \subseteq \text{Int}(\text{Cl}(\text{Int}(A, E)))\).
(v) a soft b-open \([5]\) if \((A, E) \subseteq \text{Cl}(\text{Int}(\text{Int}(A, E)))\).
(vi) a soft pre-open \([5]\) if \((A, E) \subseteq \text{Int}(\text{Cl}(A, E))\).
(vii) a soft semi-generalized closed (sg-closed) \([3]\) if \(\text{Cl}(A, E) \subseteq (U, E)\) whenever \((A, E) \subseteq (U, E)\) and \((U, E)\) is semi-open in \((X, \tau, E)\).

\(\text{(viii) a soft } \beta\text{-open} \([1]\) set if \((A, E) \subseteq \text{Cl}(\text{Int}(\text{Cl}(A, E))))\).
\(\text{(ix) a soft generalized } \beta\text{-closed (soft g}\beta\text{-closed) \([1]\) in a soft topological space } (X, \tau, E)\text{ if } \beta \text{Cl}(A, E) \subseteq (U, E)\text{ whenever } (A, E) \subseteq (U, E)\text{ and } (U, E)\text{ is soft open in } X\).
\(\text{(x) a soft generalized-semi closed (gs-closed) \([5]\) if } \text{Cl}(A, E) \subseteq (U, E)\text{ whenever } (A, E) \subseteq (U, E)\text{ and } (U, E)\text{ is open in } (X, \tau, E)\).

The complement of the soft semi open, soft regular open, soft a-open, soft b-open, soft pre-open sets are their respective soft semi closed, soft regular closed, soft } \alpha\text{-closed, soft b-closed and soft pre-closed sets.}

**Definition 2.11:** [6] A soft topological space \(X\) is called a soft } \alpha\text{-space if every soft g-closed set is soft closed in } X.

**Definition 2.12:** [5] The soft regular closure of \((A, E)\) is the intersection of all soft regular closed sets containing \((A, E)\). That is the smallest soft regular closed set containing \((A, E)\) and is denoted by \(\text{scrl}(A, E)\).

The soft regular interior of \((A, E)\) is the union of all soft regular open sets contained in \((A, E)\) and is denoted by \(\text{srnt}(A, E)\).

**Proposition 2.13:** [4] Let \((X, \tau, E)\) be a soft topological space over \(X\) and \((F, E)\) and \((G, E)\) be a soft set over \(X\). Then
1. \(\text{Int}(\text{Int}(F, E)) = \text{Int}(F, E)\)
2. \((F, E) \subseteq (G, E)\) implies \(\text{Int}(F, E) \subseteq \text{Int}(G, E)\)
3. \(\text{Cl}((\text{Cl}(F, E))) = \text{Cl}(F, E)\)
4. \((F, E) \subseteq (G, E)\) implies \(\text{Cl}(F, E) \subseteq \text{Cl}(G, E)\).

**Definition 2.14:** [8] A subset \(A\) in a topological space is defined to be a } \alpha\text{-set if } \text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(\text{Int}(A))

3. Soft sg-Closed Sets
In this section, we define new class of sets called soft sg-closed sets and establish its relationship with other soft sets is discussed.

**Definition 3.1:** A soft subset \((A, E)\) of a soft topological space \(X\) is called soft sg-closed in \(X\) if \(\text{scrl}(A, E) \subseteq (U, E)\) whenever \((A, E) \subseteq (U, E)\) and \((U, E)\) is soft semi-open in \(X\).

**Theorem 3.2:**
1. Every soft closed set is soft sg-closed.
2. Every soft g-closed is soft sg-closed.
3. Every soft a-closed set is soft sg-closed.
4. Every soft sg-closed set is soft sg-b-closed.
5. Every soft sg-closed set is soft sg-b-closed.
6. Every soft semi-closed set is soft sg-closed.
7. Every soft sg-closed set is soft sg-b-closed.

**Remark 3.3:** The converse of the above theorem is not true as seen in the following example.

**Example 3.4:** Let \(X = \{a, b, c, d\}\), \(E = \{e_1, e_2\}\). Let \(F_1, F_2, F_3, F_4, F_5, F_6\) functions from \(E\) to \(P(X)\) and are defined as follows:
\(F_1(e_1) = \{a\}, F_1(e_2) = \{c\}, F_2(e_1) = \{b\}, F_2(e_2) = \{d\}, F_3(e_1) = \{a, b\}, F_3(e_2) = \{c, d\}, F_4(e_1) = \{b, d\}, F_4(e_2) = \{a, d\}, F_5(e_1) = \{a, b, c\}, F_5(e_2) = \{a, b, d\}, F_6(e_1) = \{a, c\}, F_6(e_2) = \{a, c, d\}\). Then \(\tau = \{ \phi, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), \ldots \}\).
(F_5, E), (F_6, E) is a soft topology and elements in \( r \) are soft open sets.
1. The soft set \((A, E) = \{\{b\}, \{a, c\}\}\) is a sgb-closed but not soft closed.
2. The soft set \((A, E) = \{\{a\}, \{b, d\}\}\) is a sgb-closed but not soft g-closed.
3. The soft set \((A, E) = \{\{b, c\}, \{c\}\}\) is a sgb-closed but not soft \( \alpha \)-closed.
4. The soft set \((A, E) = \{\{b\}, \{c\}\}\) is a sgb-closed but not soft sg-closed.
5. The soft set \((A, E) = \{\{b, c\}, \{b, c\}\}\) is a sgb-closed but not soft semi closed.
6. The soft set \((A, E) = \{\{a\}, \{c, d\}\}\) is a soft \( g \beta \)-closed but not soft sgb- closed.
7. The soft set \((A, E) = \{\{a\}, \{c, d\}\}\) is a soft \( g \beta \)-closed but not soft sgb- closed.

**Remark 3.5**

We depict the above discussions in the following diagram.

![Diagram](image)

**Theorem 3.6:** If \((A, E)\) is soft semi-open and soft sgb-closed, then \((A, E)\) is soft \( b \)-closed.

**Proof:** Let \((A, E)\) be soft semi-open and soft sgb-closed. Let \((A, E) \subseteq (A, E)\) where \((A, E)\) is soft semi-open. Since \((A, E)\) is soft sgb-closed, we have \( scl(A,E) \subseteq (A,E)\). Then \((A, E) = scl(A,E).\) Hence \((A, E)\) is soft \( b \)-closed.

**Theorem 3.7:** Let \((A, E)\) be soft sgb-closed in \( X \). Then \( scl\ (A, E) - (A, E)\) does not contain any non-empty soft semi-closed set.

**Proof:** Let \((F, E)\) be a non-empty soft semi-closed set such that \((F, E) \subseteq scl(A,E) \subseteq (A,E)\). Since \((A, E)\) is soft sgb-closed, \((A, E) \subseteq X-(F,E)\) where \(X-(F,E)\) is soft semi-open implies \( scl(A,E) \subseteq X-(F,E)\). Hence \((F, E) \subseteq X-sclc(A,E)\). Now, \((F,E) \subseteq sclc(A,E) - (A,E) \cap (X-sclc(A,E))\) implies \((F,E) = \phi\). Which is a contradiction. Therefore, \( sclc(A,E) - (A,E)\) does not contain any non-empty soft semi-closed set.

**Corollary 3.8:** Let \((A, E)\) be soft sgb-closed in \( X \). Then \((A, E)\) is soft \( b \)-closed if and only if \( scl\ (A,E) - (A,E)\) is soft semi-closed.

**Proof:** Let \((A, E)\) be soft \( b \)-closed. Then \( scl\ (A,E) = (A,E)\). This implies \( scl\ (A,E) - (A,E) = \phi\) which is soft semi-closed. Assume that \( scl\ (A,E) - (A,E)\) is soft semi-closed. Then \( scl\ (A,E) - (A,E) = \phi\). Hence, \( scl\ (A,E) = (A,E)\).

**Theorem 3.17:** For a soft subset \((A, E)\) of \( X \), the following statements are equivalent:

1. \((A, E)\) is soft semi-open and soft sgb-closed.
2. \((A, E)\) is soft regular open.

**Proof:** (1) \( \Rightarrow \) (2): Let \((A, E)\) be a soft semi-open and soft sgb-closed subset of \( X \).

Then \( sbcl(A,E) \subseteq (A,E) \). Hence \( intcl(A,E) \subseteq (A,E) \). Since \((A, E)\) is soft open, we have \((A, E)\) is soft pre-open and thus \((A, E) \subseteq intcl(A,E) \). Therefore, we have \( intcl(A,E) = (A,E) \), which shows that \((A, E)\) is soft regular open.

(2) \( \Rightarrow \) (1): Since every soft regular open set is soft semi-open then \( sbcl(A,E) = (A,E) \) and \( sbcl(A,E) \subseteq (A,E) \). Hence, \((A, E)\) is soft sgb-closed.

**Remark 3.8:** Finite union of soft sgb-closed sets need not be soft sgb-closed.

**Example 3.9:** Let the two soft sets be \( G(e_1) = \{a\}, G(e_2) = \{b,d\}\) and \( H(e_1) = \{b,c\}, H(e_2) = \{b,c\}\). Then \((G, E)\) and \((H, E)\) are soft sgb-closed sets over \( X \). But their union \((A,E) = \{\{a,b,c\}, \{d,b,c\}\}\) is not soft sgb-closed.

**Remark 3.10:** Finite intersection of soft sgb-closed sets need not be soft sgb-closed.

**Example 3.11:** Let the two soft sets be \( G(e_1) = \{b,c\}, G(e_2) = \{c\}\) and \( H(e_1) = \{b\}, H(e_2) = \{c\}\). Then \((G, E)\) and \((H, E)\) are soft sgb-closed sets over \( X \). But their intersection \((A,E) = \{\{b\}, \{c\}\}\) is not soft sgb-closed.

**Definition 3.12:** A soft topological space \( X \) is said to be soft hyperconnected if the closure of every soft open subset is \( X \).

**Theorem 3.13:** Let \((A, E)\) be a soft hyperconnected space. Then every soft sgb-closed subset of \( X \) is soft sgs-closed.

**Proof:** Assume that \( X \) is soft hyperconnected. Let \((A, E)\) be soft semi-open and let \((U,E)\) be a semi-open set containing \((A, E)\).


**Theorem 3.14:** Let \((A, E)\) be a soft sgb-closed set and soft dense in \( X \). Then \((A, E)\) is soft sgb-closed.

**Proof:** Suppose that \((A, E)\) is soft sgb-closed set and soft dense in \( X \). Let \((U, E)\) be a semi-open set containing \((A, E)\).

Then \( sbcl((A,E)) = (A,E) \cup (A,E) = (A,E) \). Hence, \((A,E)\) is soft sgb-closed.

**4. Soft sg-Open Sets**

**Definition 4.1:** A soft subset \((A, E)\) of \( X \) is called soft sg-open if its relative complement is soft sg-closed.

**Lemma 4.2:** Let \((F, A)\) be a soft subset of a topological space \( X \), then \( sbint(X-(F,A)) = (X-sbint(F,A))\).

**Proof:** Let \( x \in X- sbint(F,A) \). Then \( x \notin sbint(F,A) \). That is every soft \( b \)-open set \((G, A)\) containing \( x \) is such that \((G, A) \not\subseteq (F, A) \). Hence every soft \( b \)-open set \((G, A)\) containing \( x \) intersects \( X-(F, A) \). This implies \( x \in sclc(X-(F,A)) \). Hence \( X-sbint((F,A)) \subseteq sbcl((X-(F,A))) \).

Conversely, Let \( x \in sclc(X-(F,A)) \). Then \( x \notin sbint(F,A) \). This implies \( x \notin sbint(F,A) \). Thus \( sbcl(X-(F,A)) \subseteq X-sbint((F,A)) \). Hence \( sbcl(X-(F,A)) = (X-sbint(F,A)) \).
Theorem 4.3: The soft subset \((A,E)\) of \(X\) is soft sgb-open if and only if \(F \subseteq \text{sbtint}(A,E)\) whenever \((F,E)\) is soft semi-closed and \((F,E) \subseteq (A,E)\).

Proof: Necessity: Let \((A,E)\) be soft sgb-open. Let \((F,E)\) be soft semi-closed and \((F,E) \subseteq (A,E)\). Then \(X-(A,E) \subseteq X-(F,E)\) where \(X-(F,E)\) is soft semi-open. By assumption, \(sbcl(X-(A,E)) \subseteq X-(F,E)\). By Lemma 4.2, \(X-\text{sbtint}(A,E)\) \(\subseteq X-(F,E)\). Thus \((F,E) \subseteq \text{sbtint}(A,E)\).

Sufficiency: Suppose \((F,E)\) is soft semi-closed and \((F,E) \subseteq (A,E)\) such that \((F,E) \subseteq \text{sbtint}(A,E)\). Let \((F,E) \subseteq (U,E)\) where \((U,E)\) is soft semi-open. Then \(X-(U,E) \subseteq (A,E)\) where \(X-(U,E)\) is soft semi-closed. By hypothesis, \(X-(U,E) \subseteq \text{sbint}(A,E)\). That is \(X-(A,E) \subseteq (U,E)\). Hence \(sbcl(X-(A,E)) \subseteq (U,E)\). Thus \(X-(A,E)\) is soft sgb-closed and \((A,E)\) is soft sgb-open.

Theorem 4.4: If \(\text{sbtint}(A,E) \subseteq (B,E) \subseteq (A,E)\) and \((A,E)\) is soft sgb-open then \((B,E)\) is soft sgb-open.

Proof: Let \(\text{sbtint}(A,E) \subseteq (B,E) \subseteq (A,E)\). Then \(X-(A,E) \subseteq X-(B,E)\) \(\subseteq \text{sbcl}(X-(A,E))\) by Lemma 4.2. Since \(X-(A,E)\) is soft sgb-closed, by Theorem 4.3, \((X-(A,E)) \subseteq (X-B) \subseteq \text{sbcl}(X-(A,E))\) implies \((X-(B,E)) = \text{soft sgb-closed}\). Hence \((B,E)\) is soft sgb-open.

6. Soft sgb-T1/2 Spaces

Definition 5.1: A soft topological space \(X\) is called
(i) soft sgb-T1/2 space if every soft sgb-closed set is soft b-closed.
(ii) soft sgb-space if every soft sgb-closed is soft closed.
(iii) soft sgb-T1/2 space if every soft sgb-closed set is soft b-closed. X is soft sgb-T1/2 space.

5. Soft sgb-T1/2 Spaces

Definition 5.1: A soft topological space \(X\) is called
(i) soft sgb-T1/2 space if every soft sgb-closed set is soft b-closed.
(ii) soft sgb-space if every soft sgb-closed is soft closed.
(iii) soft sgb-T1/2 space if every soft sgb-closed set is soft b-closed.

Theorem 5.2: For a soft topological space \((X, \tau, E)\) the following are equivalent
(i) \(X\) is soft sgb-T1/2 space.
(ii) Every singleton set is either soft semi-closed or soft b-open.

Proof: To prove \((i) \Rightarrow (ii)\): Let \((A,E)\) be a soft sgb-T1/2 space. Then \(X-(A,E)\) is soft semi-closed. Hence \(X-(A,E)\) is soft b-closed. Therefore, \((A,E)\) is soft b-open.

To prove \((ii) \Rightarrow (i)\): Assume that every singleton of \(X\) is either soft semi-closed or soft b-open. Let \((A,E)\) be a soft b-closed set. Then \(A \subseteq \text{sbtint}(A,E)\). Hence \((A,E)\) is soft b-open.

Theorem 5.3: For a soft topological space \((X, \tau, E)\) the following are equivalent
(i) \(X\) is soft sgb-T1/2 space.
(ii) For every soft subset \((A,E)\) of \(X\), \((A,E)\) is soft sgb-open if and only if \((A,E)\) is soft b-open.

Proof: \((i) \Rightarrow (ii)\): Let the soft topological space \(X\) be soft sgb-T1/2 and let \((A,E)\) be a soft sgb-open soft subset of \(X\). Then \(X-(A,E)\) is soft sgb-closed and \(X-(A,E)\) is soft b-closed. Hence \((A,E)\) is soft b-open.

Conversely, Let \((A,E)\) be a soft b-open subset of \(X\). Thus \(X-(A,E)\) is soft b-closed. Since every soft b-closed set is soft sgb-closed then \(X-(A,E)\) is soft sgb-closed. Hence \((A,E)\) is soft sgb-open.

Theorem 5.4: \(\text{SBO}(X, \tau, E) \subseteq \text{SSGBO}(X, \tau, E)\).

Proof: Let \((A,E)\) be soft b-open, then \(X-(A,E)\) is soft b-closed so \(X-(A,E)\) is soft sgb-closed. Thus \((A,E)\) is soft sgb-open. Hence \(\text{SBO}(\tau) \subseteq \text{SSGBO}(\tau)\).

6. References