Sylow p-subgroups and its applications

Dr. Amod Kumar Mishra

Abstract

Let p be a prime st. \( p^n \) devises order of a group G and \( p^{n+1} \) does not divide it G. s.t. \( O(H) = p^n \) is called a sylow p-subgroup of G. there are three postulates of Sylow p-subgroups first theorem shows the existence of a sylow p-subgroup of G for every prime p dividing order of G while. Second theorem shows that any two sylow p-subgroups of G are conjugate and the third theorem says about the number of sylow p-subgroups of G.

Keywords: Conjugate p-subgroups \( O(G) \), \( O(H) \).

Introduction

Let \( p \) be a prime and \( m, a+ve integer \) St. \( p^n\divides O(G) \). Then a subgroup \( H \) of G. St \( O(H) = p^n \) is called a sylow p-subgroup of G. any two sylow p-subgroups of a finite group G are conjugate to one another.

Let \( p \) be sylow p-subgroups of G is of the form \( 1+kp \) where \( 1+kp\divides O(G) \), \( K \) being a non-negative integer Let \( p \) be a sylow p-subgroups of G. Then the number of sylow p-subgroups of G is eqal to \( \left( \frac{P}{N(P)} \right) \) of G is of the form 1+kp where 1+kp\divides O(G), \( k \) being a none negative integer.

Note: If \( O(G) = p^nq \), \( (p, q)=1 \) then the number of sylow p-subgroups is 1+kp where \( 1+kp\divides p^nq \).

Example: Let G be a group of order 231. Show that N–Sylon subgroup of G is contained in the centre of G.

Soln: Let \( O(G) = 231 = 3\times7\times11 \)
The number of sylow 11–subgroup of G is \( 1+11k \) and \( (1+11k)/21 \). Clearly then it is possible if \( k=0 \) so, sylon 11-subgroup H of G is normal in G.
The number of sylo 7–subgroup of G is \( 1+7k' \) and \( (1+7k')/33 \). So \( k'=0 \), thus, Sylo 7-subgroup k of G is normal in G \( O(H) = 11, O(k) = 7 \)

Now \( O\left(\frac{G}{k}\right) = \frac{3\times7\times11}{7} = 3\times11 \) and \( 3\times (11–1) \), Thus \( \frac{G}{k} \) is cyclic, [If \( O(G) = pq \), where \( p, q \) are distinct primes, \( p,q \), \( pxq–1 \), then G is cyclic]. Thus \( \frac{G}{k} \) is cyclic and so \( G \) is abelian. But \( G \) is the smallest subgroup of G such that \( G/H \) is abelian (\( G \) denote the commutator subgroup of G).

\( \therefore \) \( G\subseteq k \Rightarrow O(G) = 1 \) or \( 7 \) if \( O(G) = 1 \) Then \( G' \) [\( e \) \( \Rightarrow x^1y^1xy = e \Rightarrow xy = yx \) for all x, y \( \epsilon \) G \( \Rightarrow \) G is abelian \( \Rightarrow G = Z(G) \Rightarrow H\subseteq Z(G) \). Let \( O(G) = 7 \Rightarrow G' = K. \) Clearly H\cap k = \{e\} as O (H\cap k) divides O (H) = 11 and O (k) = 7. Let x \( \epsilon \) H, Y \( \epsilon \) G Then \( x^1y^1xy \epsilon G' = k. \) also \( x^1y^1xy = x^{-1}(y^{-1}xy) \epsilon H \) as H is normal is G

\( \therefore \) \( x^{-1}y^{-1}xy \epsilon H\cap k = \{e\} \Rightarrow xy = yx, y \epsilon G, x \epsilon H \Rightarrow H\subseteq Z(G). \)

If \( p \) is sylow p-subgroup of G Let X \( \epsilon \) N (p) st \( O(x) = p \), then X \( \epsilon \) p. If every p-subgroup of a finite group is contained in some sylow p-subgroup of G.

\( \Rightarrow 1+kp|q \) as \( (1+kp, p^n) = 1 \)
of $p$ is the only sylow $p$-subgroups of $G$ then $p$ is normal in $G$ and if $p$ is normal $G$ then $p$ is only sylow $p$-subgroups of $G$. If $p$ is sylow $p$-subgroups of $G$ to very $p$-Subgroup of a finite group is contained in some sylow $p$-subgroups of $G$.

**Aim**

In a finite group $G$ no sylow $p$-subgroups can be properly contained in a $p$-subgroup. If let $G$ be a finite group and $p$ be a $p$-subgroup of $G$, then $p$ is sylow $p$-subgroups of $G$ if and only of no $p$-subgroup of $G$ properly contains $p$. If $O(G) = p$, $q$, where $p$, $q$ are distinct prime, $p < q$, $p|q-1$ then it is cyclic. Using sylow's theorem we can find $\neg p-1=1 \mod p$ for every prime $p$. Let $p$ we a prime dividing $O(G)$ and if $(ab)^p = a^p b^p$ for all $a$, $b \in G$. then sylow $p$-subgroups $p$ is normal in $G$ and \normal subgroup $N$ of $G$. St. $P \cap N = \{e\}$ and $G = PN$ together with $G$ has non trivial centre.

**Conclusion**

Let $G$ be a finite group and $H \leq G$ suppose $p$ is a prime dividing $O(G)$. Let $p$ be a sylow $p$-subgroups of $H$ contained in some sylow $p$-subgroups of $G$. then $p = S \cap H$. Let $G$ be a finite group and let $H$ be normal in $G$. If $p$ be a prime dividing $O(G)$. If $(|G:H|, p)=1$ Then $H$ contains every sylow $p$-subgroups of $G$ for finite group $G$ and $p$ being a prime dividing $O(H)$ where $H \leq G$ then the number of sylow $p$-subgroups of $H$ is less than or equal to the number of sylow $p$-subgroups of $G$.

Let $p$ be a prime dividing $O(G)$ St. if $K$ is normal in $G$ and $p$ is a sylow $p$-subgroups of $G$. Then $p \cap K$ is a sylow $p$-subgroups of $G$ and $\frac{pK}{K}$ is a sylow $p$-subgroups of $G/K$.

Every sylow $p$-subgroups of $\frac{G}{K}$ is of the form $\frac{pK}{K}$ where $p$ is a sylow $p$-subgroup of $G$. Let $G$ be a group of order $p q r$, $p < q < r$ being a prime. In this situation some sylow $p$-subgroup of $G$ is normal in $G$ and then $G$ is can’t be simple.

**References**