On Nano generalized\(^{\wedge}\)-closed and open sets in Nano topological spaces

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Abstract

This paper focuses on nano generalized\(^{\wedge}\)-closed sets and nano generalized\(^{\wedge}\)-open sets in nano topological spaces and certain properties of these are investigated. Also the converse of the theorems are proved with examples.

Keywords: Nano topology, Nano closed sets, Nano open sets, Nano interior, Nano closure, Nano generalized closed sets, Nano generalized\(^{\wedge}\)-closed sets.

1. Introduction

In 1970, Levine [4] introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. This concept was introduced as a generalization of closed sets in Topological spaces through which new results in general topology were introduced. Lellis Thivagar [3] introduced Nano topological space with respect to a subset X of a universe which is defined in terms of lower and upper approximations of X. The elements of Nano topological space are called Nano open sets. He has also defined Nano closed sets, Nano-interior and Nano closure of a set. He also introduced the weak forms of Nano open sets namely Nano-a open sets, Nano semi open sets and Nano pre-open sets.

In this paper, we define a new class of sets called nano generalized\(^{\wedge}\)-closed and open sets in nano topological spaces and study the relationships with other nano sets.

2. Preliminaries

Definition 2.1 [6]: A g\(^{\wedge}\)-closed set [20] if cl(A)\(\subseteq\)G whenever A\(\subseteq\)G and G is semi-open in (X, \(\tau\)). The complement of a g\(^{\wedge}\)-closed set is called a g\(^{\wedge}\)-open set.

Definition 2.2: [2]

Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U,R) is said to be the approximation space. Let U be a non-empty finite set of objects called the universe and R be an equivalence relation. Let X \(\subseteq\) U.

1. The lower approximation of X with respect to R is the set of all objects, which can be for certain classified as X with respect to R and it is denoted by L\(_R\)(X)

2. The upper approximation of X with respect to R is the set of all objects, which can be possibly classified as X with respect to R and it is denoted by U\(_R\)(X).That is,

3. The boundary region of X with respect to R is the set of all objects, which can be classified neither as X nor as not-X with respect to R and it is denoted by, B\(_R\)(X). That is,

Property 2.3: [2] If (U, R) is an approximation space and X, Y \(\subseteq\) U, then

1. L\(_R\)(X) \(\subseteq\) X \(\subseteq\) U \(_R\)(X)
2. L\(_R\)(\(\emptyset\)) = U \(_R\)(\(\emptyset\)) and U \(_R\)(U) = \(_R\)(U) = U
(iii) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$
(iv) $U_R(X \cap Y) \subset U_R(X) \cap U_R(Y)$
(v) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$
(vi) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$

Definition 2.6: \[2\]
Let $U$ be the universe, $R$ be an equivalence relation on $U$ and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ whenever $X \subseteq U$. Then by property 2.3 $\tau_R(X)$ satisfies the following axioms:
1. $U$ and $\emptyset$ belongs to $\tau_R(X)$
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$
3. The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$. That is, $\tau_R(X)$ is a topology on $U$ called the Nano topology on $U$ with respect to $X$. We call $(U, \tau_R(X))$ as the Nanotopological space. The elements of $\tau_R(X)$ are called as Nano-open sets.

Remark 2.5 \[2\] If $\tau_R(X)$ is the Nano topology on $U$ with respect to $X$, then the set $B = \{ U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

Definition 2.6: \[2\]
If $(U, \tau_R(X))$ is a Nano topological space with respect to $X$ where $X \subseteq U$ and if $A \subseteq U$, then the Nano interior of $A$ is defined as the union of all Nano-open subsets of $A$ and it is denoted by $Ncl(A)$. That is, $Ncl(A)$ is the largest Nano-open subset of $A$. The Nano closure of $A$ is defined as the intersection of all Nano closed sets containing $A$ and it is denoted by $Nscl(A)$. That is, $Ncl(A)$ is the smallest Nano closed set containing $A$.

Definition 2.7: \[2\]
Let $(U, \tau_R(X))$ be a Nano topological space. A subset $A$ of $(U, \tau_R(X))$ is called Nano generalized closed set (briefly Ng-closed) if $Ncl(A) \subseteq V$ whenever $A \subseteq V$ and $V$ is Nano open.

Definition 2.8: \[2\]
A subset $A$ of $(U, \tau_R(X))$ is called Nano semi-pre generalized closed set (Nspg) if $Nspcl(A) \subseteq V$ whenever $A \subseteq V$ and $V$ is Nano semi-open.

Definition 2.9: \[2\]
A subset $A$ of $(U, \tau_R(X))$ is called nano semi-generalized closed set (briefly Nsg-closed) if $Nsc(A) \subseteq V$ whenever $A \subseteq V$ and $V$ is nano semi-open in $(U, \tau_R(X))$.

3. Nano generalized^\^closed sets
Throughout this paper $(U, \tau_R(X))$ is a Nano topological space with respect to $X$ where $X \subseteq U$, $R$ is an equivalence relation on $U$, $U/R$ denotes the family of equivalence classes of $U$ by $R$.

Definition 3.1
Let $(U, \tau_R(X))$ be a nano topological space. A subset $A$ of $(U, \tau_R(X))$ is called Nano generalized^\^closed set (briefly Ng^\^closed) if $Ncl(A) \subseteq U$ and if $A \subseteq U$, then the Nano interior of $A$ is defined as the union of all Nano-open subsets of $A$ and it is denoted by $Ncl(A)$.

Theorem 3.2
Every Ng-closed set is Nano semi-generalized closed set.

Proof
Let $A$ be a Ng^-closed set. Then $Ncl(A) \subseteq V$ where $A \subseteq V$ and $V$ is Nano semi-open in $U$. But $Nspcl(A) \subseteq Ncl(A)$ where $A \subseteq V$, $V$ is Nano semi-open in $U$. Now we have, $Nspcl(A) \subseteq U$ where $A \subseteq V$ and $V$ is Nano semi-open in $U$.

Hence, $A$ is Nano semi-generalized closed set.

Remark 3.3
The converse of the above theorem need not be true as shown by the example.

Example 3.4
Let $U = \{a,b,c,d\}$ with $X = \{a,c\}$ with $U/R = \{\{c\}, \{a,b\}, \{d\}\}$. Let $A = \{a,b,d\}$ be a Nano generalized-closed set. Here, $Ncl(A) \subseteq V$ where $A \subseteq V$, $V$ is Nano semi-open in $U$. Hence, $A = \{a,b,d\} \subseteq U$ is nano semi-generalized closed set. $A \subseteq \{a,b,c,d\}$, $Ncl(A) \subseteq \{a,b,c,d\}$

Which implies that $A$ is Nano semi-generalized closed, but $A$ is not a Ng^-closed.

Theorem 3.5
Every Ng^-closed set is Nano semi-pre generalized closed set.

Proof
Let $A$ be Ng^-closed set. Then $Ncl(A) \subseteq V$ where $A \subseteq V$ and $V$ is Nano semi-open in $U$. But, $Nspcl(A) \subseteq Ncl(A)$ where $A \subseteq V$, $V$ is Nano semi-open set in $U$.

Now, we have $Nspcl(A) \subseteq U$ where $A \subseteq V$ and $V$ is Nano semi-open in $U$.

Hence, $A$ is Nano semi-pre generalized closed set.

Remark 3.6
The converse of the above theorem need not be true as shown by the example.

Example 3.7
Let $U = \{a,b,c,d\}$ with $X = \{a,c\}$ with $U/R = \{\{c\}, \{a,b\}, \{d\}\}$.

Let $A = \{a,c\}$ be a Nano semi-pre generalized closed set. Here $Nspcl(A) \subseteq V$, where $A \subseteq V$ and $V$ is Nano semi-open in $U$.

Hence, $A = \{a,c\} \subseteq U$ is a Nano semi-pre generalized closed set. $A \subseteq \{a,c\}$, $Nspcl(A) \subseteq \{a,b,c,d\}$.

Which implies that $A$ is Nano semi-pre generalized. But, $A$ is not a Ng^-closed set.

Theorem 3.8
Every Ng-closed is Ng^-closed set.

Proof
Let $A$ be Ng-closed set. Then $Ncl(A) \subseteq U$, where $A \subseteq V$ and $V$ is Nano open in $U$. Since, every Nano open is Nano semi-open in $U$. Now, we have $Ncl(A) \subseteq V$, where $A \subseteq V$ and $V$ is semi open in $U$. 
Hence, A is $\text{Ng}^\sim$-closed set.

**Remark 3.9**

The converse of the above theorem need not be true as shown by the example.

**Example 3.10**

Let $U=\{a,b,c,d\}$ with $U/R = \{\{a\},\{c\},\{b,d\}\}$ and $X=\{a,b\}$
Let $A=\{b,c\}$ be a $\text{Ng}^\sim$-closed set.

Here $\text{Ncl}(A)\subseteq V$, where $A\subseteq V$ and $V$ is Nano semi-open in $U$.

Hence, $A=\{b,c\}$ be a $\text{Ng}^\sim$-closed set.

**Theorem 3.11**

Any Nano closed set is $\text{Ng}^\sim$-closed set.

**Proof**

Let $A\subseteq V$ and $V$ is Nano semi-open in $\tau_R(X)$.

Since, $A$ is Nano closed, $\text{Ncl}(A)\subseteq A$.

That is, $\text{Ncl}(A)\subseteq A\subseteq V$.

Hence, $A$ is $\text{Ng}^\sim$-closed set.

**Remark 3.12**

The converse of the above theorem need not be true as shown by the example.

**Example 3.13**

Let $u=\{a,b,c,d\}$ with $X=\{a,c\}$ with $U/R=\{\{c\},\{a,b\},\{d\}\}$.

$\tau_R(X) = \{U, \emptyset, \{c\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$

Nano closed sets are $\{\emptyset, U, \{a,b,d\}, \{d\}, \{c,d\}\}$

Let $A=\{a,d\}$ and $A\subseteq \{a,b,c,d\}$, $\text{Ncl}(A)\subseteq \{a,b,c,d\}$

Which implies that, A is $\text{Ng}^\sim$-closed.

But, A is not a Nano closed.

**Theorem 3.15**

Let $U=\{a,b,c,d\}$ with $U/R = \{\{c\},\{a,c\},\{d\}\}$ and $X=\{a,c\}$.

Then the Nanotopology $\tau_R(X) = \{U, \emptyset, \{c\}, \{a,b,c\}, \{a,b\}\}$.

Nano closed sets are $\{\emptyset, U, \{a,b,d\}, \{d\}, \{c,d\}\}$

Let $V=\{c,d\}$ and $A=\{c\}$.

Then $\text{Ncl}(A)=\{c,d\} \subseteq V$.

That is A is said to be $\text{Ng}^\sim$-closed in $(U, \tau_R(X))$.

**Theorem 3.16**

A subset $A$ of $(U, \tau_R(X))$ s $\text{Ng}^\sim$-closed if $\text{Ncl}(A)-A$ contains no nonempty $\text{Ng}^\sim$-closed set.

**Proof**

Suppose if $A$ is $\text{Ng}^\sim$-closed. Then $\text{Ncl}(A) \subseteq V$ where $A\subseteq V$ and $V$ is semi-open.

Let $Y$ be nano closed subset of $\text{Ncl}(A)-A$. Then $A\subseteq Y$ and $Y$ is Nano semi-open.

Since $A$ is $\text{Ng}^\sim$-closed, $\text{Ncl}(A)\subseteq Y$ implies that $Y\subseteq \emptyset$.

Therefore, $Y$ is empty.

**Theorem 3.17**

If $A$ and $B$ are $\text{Ng}^\sim$-closed, then $A \cup B$ is $\text{Ng}^\sim$-closed.

**Proof**

Let $A$ and $B$ are $\text{Ng}^\sim$-closed sets.

Then $\text{Ncl}(A)\subseteq V$ where $A\subseteq V$ and $V$ is Nano semi-open and $\text{Ncl}(B)\subseteq V$ where $B\subseteq V$ and $V$ is Nano semi-open.

Since, $A$ and $B$ are subsets of $V$, $(A \cup B)$ is a subset of $V$ and $V$ is Nano semi-open.

Then, $\text{Ncl}(A)\cup \text{Ncl}(B) \subseteq V$ which implies that $(A \cup B)$ is $\text{Ng}^\sim$-closed.

**Remark 3.18**

The intersection of two $\text{Ng}^\sim$-closed is again an $\text{Ng}^\sim$-closed set.

**Proof**

Let $A$ and $B$ are $\text{Ng}^\sim$-closed sets.

Then $\text{Ncl}(A)\subseteq V$ where $A\subseteq V$ and $V$ is Nano semi-open and $\text{Ncl}(B)\subseteq V$ where $B\subseteq V$ and $V$ is Nano semi-open.

Since, $A$ and $B$ are subsets of $V$, $(A \cap B)$ is a subset of $V$ and $V$ is Nano semi-open.

Then, $\text{Ncl}(A)\cap \text{Ncl}(B) \subseteq V$ which implies that $(A \cap B)$ is $\text{Ng}^\sim$-closed.

**Theorem 3.19**

If $A$ is $\text{Ng}^\sim$-closed and $A\subseteq B \subseteq \text{Ncl}(A)$, then $B$ is $\text{Ng}^\sim$-closed.

**Proof**

Let $B\subseteq V$ where $V$ is Nano semi-open in $\tau_R(X)$.

Then $A\subseteq B$ implies $A\subseteq V$.

Since $A$ is $\text{Ng}^\sim$-closed, $\text{Ncl}(A)\subseteq V$ also $B\subseteq \text{Ncl}(A)$ implies $\text{Ncl}(B)\subseteq \text{Ncl}(A)$.

Thus $\text{Ncl}(B)\subseteq V$ and so $B$ is $\text{Ng}^\sim$-closed.

**Theorem 3.20**

A $\text{Ng}^\sim$-closed set $A$ is Nano closed if and only if $\text{Ncl}(A)-A$ is Nano closed.

**Proof**

(Necessary part)

Let $A$ be Nano closed.

Then $\text{Ncl}(A)-A=\emptyset$ which is Nano closed.
Suppose $Ncl(A)-A$ is Nano closed.
Then $Ncl(A)-A=\emptyset$.
Since $A$ is Nano closed. That is $Ncl(A)=A$ or $A$ is Nano closed.

**Theorem 3.21**
Suppose that $B \subseteq A \subseteq U$, $B$ is an Ng^-closed set relative to $A$ and that $A$ is Ng^-closed subset of $U$. Then $B$ is Ng^-closed relative to $U$.

**Proof**
Let $B \subseteq V$ and suppose that $V$ is Nano semi-open in $U$. Then $B \subseteq A \cap V$.
Therefore $Ncl(B) \subseteq A \cap V$ and $A \subseteq V \cup Ncl(B)$.
It follows that $A \cap Ncl(B) \subseteq A \cap V$ and $A \subseteq V \cup Ncl(B)$.
Since, $A$ is Ng^-closed in $U$, we have $Ncl(A) \subseteq V \cup Ncl(B)$.
Therefore $Ncl(B) \subseteq Ncl(A) \subseteq Ncl(B)$ and so $Ncl(B) \subseteq V$.
Then, $B$ is Ng^-closed relative to $V$.

**Corollary 3.22**
Let $A$ be Ng^-closed set and suppose that $F$ is a Nano closed set. Then $A \cap F$ is an Ng^-closed set which is given in the following example.

**Example 3.23**
Let $U=\{a,b,c,d\}$ with $X=\{a,c\}$ with $U/R=\{\{c\},\{a,b\}\}$.
Then $\tau_R(X)=\{U,\emptyset,\{c\},\{a,b,\},\{a,b,c\}\}$
The Nano closed sets are $\emptyset, U, \{a,b,d\}, \{d\}, \{c,d\}$
Let $A=\{a,d\}$ and $B=\{b,d\}$.
Then $A \cap B=\{d\}$ is an Ng^-closed set.

**Theorem 3.24**
For each $a \in U$, either $\{a\}$ is Nano closed or $\{a\}^c$ is Nano generalized^-closed set in $\tau_R(X)$.

**Proof**
Suppose $\{a\}$ is not Nano closed in $U$.
Then $\{a\}^c$ is not Nano open and the only Nano open set containing $\{a\}^c$ is $V \subseteq U$.
That is $\{a\}^c \subseteq U$. Therefore $Ncl(\{a\}^c) \subseteq U$.
Which implies $\{a\}^c$ is Ng^-closed set in $\tau_R(X)$.

4. Nano generalized^-open sets

**Definition 4.1**
A subset $A$ is Nano topological space $U$ is called Nano generalized^-open (simply Ng^-open) if $A^c$ is Nano g^-closed,

**Example 4.2**
Let $U=\{a,b,c,d\}$ with $U/R=\{\{c\},\{a,b\}\}$ and $X=\{a,c\}$
Then, the Nano topology is defined as,
$\tau_R(X)=\{U,\emptyset,\{c\},\{a,b,c\},\{a,b\}\}$.
Let $A=\{a,b\}$, then $A^c=\{c,d\}$ is a Ng^-closed, since $U$ is the only Nano semi open set containing $A^c$.
Therefore, $A$ is Ng^-open.
If $A=\{a,c,d\}$ then $A^c=\{b\}$ is not Ng^-closed.
Since $Ncl(A^c)=Ncl(\{b\})=\{a,b,d\}$ and $\{a,b,d\}\not\subseteq \{b\}$.
A semi open set such that $A^c \subseteq V$.
Therefore $A$ is not Ng^-open.

**Theorem 4.3**
If $A$ and $B$ are Ng^-open. Then $A \cap B$ is Ng^-open.