MA Gopalan, S Vidhyalakshmi and Presenna Ramanand

Abstract
The binary quadratic equation represented by the positive pellian \( y^2 = 32x^2 + 36 \) is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas.

Keywords: Binary quadratic, hyperbola, parabola, pell equation, integral solutions

Introduction
The binary quadratic equation of the form \( y^2 = Dx^2 + 1 \) where \( D \) is non-square positive integer has been studied by various mathematicians for its non-trivial integer solutions when \( D \) takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by \( y^2 = 32x^2 + 36 \) is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of analysis
The diophantine equation under consideration is
\[
y^2 = 32x^2 + 36
\]
(1)
The smallest positive integer solution \( (x_0, y_0) \) of (1) is
\[
x_0 = 3, \quad y_0 = 18
\]
To obtain the other solutions of (1), consider the Pell equation
\[
y^2 = 32x^2 + 1
\]
(2)
whose smallest positive integer solution is
\[
x_0 = 3, \quad y_0 = 17
\]
The general solution \( (\bar{x}_n, \bar{y}_n) \) of (2) is given by
\[
\bar{y}_n + \sqrt{32} \bar{x}_n = (17 + 3\sqrt{32})^{n+1}, \quad \text{where} \quad n = 0, 1, 2, \ldots
\]
(3)
Since irrational roots occur in pairs, we have
\[
\bar{y}_n - \sqrt{32} \bar{x}_n = (17 - 3\sqrt{32})^{n+1}, \quad \text{where} \quad n = 0, 1, 2, \ldots
\]
(4)
From (3) and (4), solving for \( (\bar{x}_n, \bar{y}_n) \), we have
\[
\bar{y}_n = \frac{1}{2} f_n, \quad \bar{x}_n = \frac{1}{2\sqrt{32}} g_n
\]
where
\[
f_n = (17 + 3\sqrt{32})^{n+1} + (17 - 3\sqrt{32})^{n+1}, \quad g_n = (17 + 3\sqrt{32})^{n+1} - (17 - 3\sqrt{32})^{n+1}
\]
Applying Brahmagupta Lemma between the solutions \((x_0, y_0)\) and \((\bar{x}_n, \bar{y}_n)\), the other integer solutions to (1) are given by

\[ 2\sqrt{32}x_{n+1} = 3\sqrt{32}f_n + 18g_n \quad 2y_{n+1} = 18f_n + 3\sqrt{32}g_n \]

A few numerical examples are given in the Table: 1 below

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x_{n+1})</th>
<th>(y_{n+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>0</td>
<td>105</td>
<td>594</td>
</tr>
<tr>
<td>1</td>
<td>3567</td>
<td>20178</td>
</tr>
<tr>
<td>2</td>
<td>121173</td>
<td>685458</td>
</tr>
</tbody>
</table>

A few interesting relations among the solutions are given below

- \[ 68y_{n+2} - 2y_{n+3} - 2y_{n+1} = 0 \]
- \[ x_{n+3} - 34x_{n+2} + x_{n+1} = 0 \]
- \[ 3y_{n+1} - x_{n+2} + 17x_{n+1} = 0 \]
- \[ 3y_{n+2} - 17x_{n+2} + x_{n+1} = 0 \]
- \[ 3y_{n+3} - 577x_{n+2} + 17x_{n+1} = 0 \]
- \[ 102y_{n+1} - x_{n+3} + 577x_{n+1} = 0 \]
- \[ 6y_{n+2} - x_{n+3} + x_{n+1} = 0 \]
- \[ 17y_{n+1} - y_{n+2} + 96x_{n+1} = 0 \]
- \[ 17y_{n+3} - 577y_{n+2} - 96x_{n+1} = 0 \]
- \[ 577x_{n+3} - 102y_{n+3} - x_{n+1} = 0 \]
- \[ 577y_{n+1} - y_{n+3} + 3264x_{n+1} = 0 \]
- \[ 577y_{n+2} + 289y_{n+3} + 1829568x_{n+1} = 0 \]
- \[ 3y_{n+1} - 17x_{n+3} + 577x_{n+2} = 0 \]
- \[ 3y_{n+2} - x_{n+3} + 17x_{n+2} = 0 \]
- \[ 3y_{n+3} - 17x_{n+3} + x_{n+2} = 0 \]
- \[ 17y_{n+2} - y_{n+3} + 96x_{n+2} = 0 \]
- \[ y_{n+3} - y_{n+1} - 192x_{n+2} = 0 \]
- \[ y_{n+3} - 17y_{n+2} - 96x_{n+2} = 0 \]
- \[ 577y_{n+2} - 17y_{n+3} - 96x_{n+3} = 0 \]
- \[ 17y_{n+3} - y_{n+2} - 96x_{n+3} = 0 \]
- \[ y_{n+1} - 577y_{n+3} + 3264x_{n+3} = 0 \]
- \[ y_{n+3} - 34y_{n+2} + y_{n+1} = 0 \]

Each of the following expressions represents a cubical integer:

- \[ \frac{1}{108}((36x_{3n+4} - 1188x_{3n+3}) + 3(36x_{n+2} - 1188x_{n+1})) \]
- \[ \frac{1}{306}((3x_{3n+5} - 3363x_{3n+3}) + 3(3x_{n+3} - 3363x_{n+1})) \]
- \[ \frac{1}{102}((6y_{3n+4} - 1120x_{3n+3}) + 3(6y_{n+2} - 1120x_{n+1})) \]
- \[ \frac{1}{1731}((3y_{3n+5} - 19024x_{3n+3}) + 3(3y_{n+3} - 19024x_{n+1})) \]
Each of the following expressions represents a bi-quadratic integer:

\[ \frac{1}{108} \left( 36x_{2n+3} - 1188x_{4n+2} + 4(36x_{2n+3} - 1188x_{2n+2} + 216) - 216 \right) \]

\[ \frac{1}{306} \left( 2x_{2n+3} - 3363x_{4n+4} + 4(3x_{2n+3} - 3363x_{2n+2} + 612) - 612 \right) \]

\[ \frac{1}{102} \left( 6y_{4n+4} - 1120x_{4n+2} + 4(6y_{2n+3} - 1120x_{2n+2} + 204) - 204 \right) \]

\[ \frac{1}{1731} \left( 3y_{4n+6} - 19024x_{4n+4} + 4(3y_{2n+4} - 19024x_{2n+2} + 3462) - 3462 \right) \]

\[ \frac{1}{9} \left( 39x_{4n+4} - 3363x_{4n+5} + 4(99x_{2n+4} - 3363x_{2n+3} + 18) - 18 \right) \]

\[ \frac{1}{51} \left( 19x_{2n+5} - 16x_{4n+5} + 4(99y_{2n+2} - 16x_{2n+2} + 102) - 102 \right) \]

\[ \frac{1}{3} \left( 99x_{4n+5} - 560x_{4n+6} + 4(99y_{2n+3} - 560x_{2n+3} + 6) - 6 \right) \]

\[ \frac{1}{51} \left( 99y_{4n+6} - 19024x_{4n+5} + 4(99y_{2n+4} - 19024x_{2n+3} + 102) - 102 \right) \]

\[ \frac{1}{1731} \left( 19x_{4n+4} - 16x_{4n+6} + 4(3363y_{2n+2} - 16x_{2n+4} + 3462) - 3462 \right) \]

\[ \frac{1}{51} \left( 3363y_{4n+5} - 560x_{4n+6} + 4(3363y_{2n+3} - 560x_{2n+4} + 102) - 102 \right) \]

\[ \frac{1}{3} \left( 3363y_{4n+6} - 19024x_{4n+6} + 4(3363y_{2n+4} - 19024y_{2n+4} + 6) - 6 \right) \]

\[ \frac{1}{18} \left( 35y_{4n+4} - y_{4n+5} + 4(35y_{2n+2} - y_{2n+3} + 36) - 36 \right) \]

\[ \frac{1}{612} \left( 1189y_{4n+4} - y_{4n+6} + 4(1189y_{2n+2} - y_{2n+4} + 1224) - 1224 \right) \]
Each of the following expressions represents a Nasty Number

\[
\frac{1}{18} \left( 1189y_{4n+5} - 35y_{4n+6} \right) + 4 \left( 1189y_{2n+3} - 35y_{2n+4} + 36 \right) - 36
\]

\[
\frac{1}{18} (36x_{2n+3} - 1188x_{2n+2} + 216)
\]

\[
\frac{1}{51} (3x_{2n+4} - 3363x_{2n+2} + 612)
\]

\[
\frac{1}{17} (6y_{2n+3} - 1120x_{2n+2} + 204)
\]

\[
\frac{2}{577} (3y_{2n+4} - 19024x_{2n+2} + 3462)
\]

\[
\frac{2}{3} (99x_{2n+4} - 3363x_{2n+3} + 18)
\]

\[
\frac{2}{17} (99y_{2n+3} - 16x_{2n+3} + 102)
\]

\[
2(99y_{2n+3} - 560x_{2n+3} + 6)
\]

\[
\frac{2}{17} (99y_{2n+4} - 19024x_{2n+3} + 102)
\]

\[
\frac{2}{577} (3363y_{2n+2} - 16x_{2n+4} + 3462)
\]

\[
\frac{2}{17} (3363y_{2n+3} - 560x_{2n+4} + 102)
\]

\[
2(3363y_{2n+4} - 19024x_{2n+4} + 6)
\]

\[
\frac{1}{3} (35y_{2n+2} - y_{2n+3} + 36)
\]

\[
\frac{1}{102} (1189y_{2n+2} - y_{2n+4} + 1224)
\]

\[
\frac{1}{3} (1189y_{2n+3} - 35y_{2n+4} + 36)
\]

Remarkable observations
(i) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table 2 below:

<table>
<thead>
<tr>
<th>S. No</th>
<th>Hyperbolas</th>
<th>((X_n, Y_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(32Y_n^2 - X_n^2 = 1492992)</td>
<td>((6720x_{n+1} - 192x_{n+2}, 36x_{n+2} - 1188x_{n+1}))</td>
</tr>
<tr>
<td>2</td>
<td>(32Y_n^2 - X_n^2 = 11985408)</td>
<td>((19024x_{n+1} - 16x_{n+2}, 3x_{n+3} - 3363x_{n+1}))</td>
</tr>
<tr>
<td>3</td>
<td>(32Y_n^2 - X_n^2 = 1331712)</td>
<td>((6336x_{n+1} - 32y_{n+2}, 6y_{n+2} - 1120x_{n+1}))</td>
</tr>
<tr>
<td>4</td>
<td>(32Y_n^2 - X_n^2 = 383534208)</td>
<td>((107616x_{n+1} - 16y_{n+3}, 3y_{n+3} - 19024x_{n+1}))</td>
</tr>
<tr>
<td>5</td>
<td>(32Y_n^2 - X_n^2 = 10368)</td>
<td>((19024x_{n+2} - 560x_{n+3}, 99x_{n+3} - 3363x_{n+2}))</td>
</tr>
<tr>
<td>6</td>
<td>(32Y_n^2 - X_n^2 = 332928)</td>
<td>((96x_{n+2} - 560y_{n+1}, 99y_{n+1} - 16x_{n+2}))</td>
</tr>
<tr>
<td>7</td>
<td>(32Y_n^2 - X_n^2 = 1152)</td>
<td>((3168x_{n+2} - 560y_{n+2}, 99y_{n+2} - 560x_{n+2}))</td>
</tr>
<tr>
<td>8</td>
<td>(32Y_n^2 - X_n^2 = 332928)</td>
<td>((107616x_{n+2} - 560y_{n+3}, 99y_{n+3} - 19024x_{n+2}))</td>
</tr>
<tr>
<td>9</td>
<td>(32Y_n^2 - X_n^2 = 383534208)</td>
<td>((96x_{n+3} - 19024y_{n+1}, 3363y_{n+1} - 16x_{n+3}))</td>
</tr>
</tbody>
</table>
(ii) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below:

### Table 3: Parabolas

<table>
<thead>
<tr>
<th>S. No</th>
<th>Parabolas</th>
<th>((X'_n, Y'_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3456,1492992</td>
<td>((6720x_{n+1} - 192x_{n+2}), (36x_{2n+3} - 1188x_{2n+2} + 216))</td>
</tr>
<tr>
<td>2</td>
<td>9792,11985408</td>
<td>((19024x_{n+1} - 16x_{n+3}), (3x_{2n+4} - 3363x_{2n+2} + 612))</td>
</tr>
<tr>
<td>3</td>
<td>3264,1331712</td>
<td>((6336x_{n+1} - 32y_{n+2}), (6y_{2n+3} - 1120x_{2n+2} + 204))</td>
</tr>
<tr>
<td>4</td>
<td>55392,383534208</td>
<td>((107616x_{n+1} - 16y_{n+3}), (3y_{2n+4} - 19024x_{2n+2} + 3462))</td>
</tr>
<tr>
<td>5</td>
<td>288,10368</td>
<td>((99x_{2n+4} - 3363x_{2n+3} + 18))</td>
</tr>
<tr>
<td>6</td>
<td>1632,332928</td>
<td>((96x_{n+2} - 560y_{n+1}), (99y_{2n+2} - 16x_{2n+3} + 102))</td>
</tr>
<tr>
<td>7</td>
<td>96,1152</td>
<td>((3168x_{n+2} - 560y_{n+2}), (99y_{2n+3} - 560y_{2n+3} + 6))</td>
</tr>
<tr>
<td>8</td>
<td>1632,332928</td>
<td>((107616x_{n+2} - 560y_{n+3}), (99y_{2n+4} - 19024x_{2n+3} + 102))</td>
</tr>
<tr>
<td>9</td>
<td>55392,383534208</td>
<td>((96x_{n+3} - 19024y_{n+1}), (3363y_{2n+2} - 16x_{2n+4} + 3462))</td>
</tr>
<tr>
<td>10</td>
<td>1632,332928</td>
<td>((3168x_{n+3} - 19024y_{n+2}), (3363y_{2n+3} - 560x_{2n+4} + 102))</td>
</tr>
<tr>
<td>11</td>
<td>96,1152</td>
<td>((107616x_{n+3} - 19024y_{n+3}), (3363y_{2n+4} - 19024x_{2n+4} + 6))</td>
</tr>
<tr>
<td>12</td>
<td>576,41472</td>
<td>((6y_{n+2} - 198y_{n+1}), (35y_{2n+2} - y_{2n+3} + 36))</td>
</tr>
<tr>
<td>13</td>
<td>19584,47941632</td>
<td>((6y_{n+3} - 6726y_{n+1}), (1189y_{2n+2} - y_{2n+4} + 1224))</td>
</tr>
<tr>
<td>14</td>
<td>576,41472</td>
<td>((198y_{n+3} - 6726y_{n+2}), (1189y_{2n+3} - 35y_{2n+4} + 36))</td>
</tr>
</tbody>
</table>

**Conclusion**

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation \(y^2 = 32x^2 + 36\). As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their integer solutions along with suitable properties.

**References**

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