On characterisation of p-solvable groups and its invariance

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Abstract
Our main aim is to derive the relationship between representation theory groups with invariants of subgroup of a finite group. We consider p-solvable groups for examining the converse cropped up due to Brauer's hypothesis. The converse of Brauer's hypothesis was proved partially for p-solvable groups by D. Gluck and T. Wolf. We show that for an arbitrary group for p-solvable groups, there exists some characteristics due to the heights of the characters in \( \text{Irr}(B) \) with respect to the exponents of the character of the group \( D \) (Dihedral group).

Keywords: Character, Brauer hypothesis, p-solvable group, irreducible, defect group invariant

Introduction
The most important objective in representation of finite group theory is to establish its connection with theory invariants of a finite group \( G \) with some local subgroups of \( G \). Let \( p \) be a prime, let \( G \) be a finite group and let \( B \) a \( p \)-block of \( G \) with defect group \( D \). If \( X \in \text{Irr}(B) \) is an irreducible complex character of \( B \), the height \( h \) of \( X \) is the non-negative integer which satisfy the given relation.

\[
\chi(1)_b = P_a - d + h \quad \ldots \quad (1.1)
\]

Where \( |G| = P^a \) and \( |D| = P^d \). If \( B \) is a Brauer's \( p \)-block of \( G \) having defect group \( D \), then complexity of \( D \) reflects in the set \( \text{Irr}(B) \) of complex irreducible characters in \( B \), which implies that \( D \) is an abelian if and only all characters in \( \text{Irr}(B) \) have height zero. F. Fong proved it for p-solvable groups. We show that the group of outer automorphisms of \( E \) that fix all the elements of \( Z(E) \), contains as a subgroup, which is either the symplectic group \( \text{Sp}_{2n}(p) \) or an orthogonal group \( \text{O}_{2n}(2) \). Let us choose \( n \) to be sufficiently large and assume that \( \text{out}(E) \) has a subgroup that is a solvable. Frobenius group \( F = PH \) whose complement \( P \) is a cyclic \( P \)-group of order \( Pm \) with \( m \geq 2 \). Let \( G \) be this extension of \( E \) by \( F \). Then \( \text{O}_p(G) = 1 \), which means that \( G \) has a unique \( p \)-block. Now, let \( \theta \in \text{Irr}(E) \) be non linear and by suitable application of \( \theta \).

Theorem
Suppose that \( A \) acts coprimely as automorphism on a finite group \( G \). If \( C_A(G) = 1 \) then there exists a nilpotent \( A \)-invariant subgroup \( H \) of \( G \) such that \( C_A(H) = 1 \).

Proof: R. Guratinich and R. Robinson proved invariance for groups. We consider here with basis of the classification of finite simple groups. The properties of heights of characters are compared with the order of the when defect group which are comparable if \( h \) is the height of an irreducible character of a block \( B \) and \( d \) is the defect of \( B \). Then \( h \leq d \). Also \( h = d \) if and only if \( d = 0 \).

If \( d > 0 \), the ratio \( \frac{h}{d} \) is sufficient condition for \( G = \text{SL}_n(p) \) with the defining characteristic \( p \), for all characters except the Steinberg character lying in the principal block. Also there exists a unipotent character with height \( \frac{(n-1)(n-2)}{2} \). But the defect of the block is \( n \frac{(n-1)}{2} \), determined it for p-solvable groups which is contradicts to assumption of the given theorem.
Theorem
Suppose the G is a solvable group. Let X∈Irr(B) of height h and B has defect group D. If |D:Z(D)| = P^n then h ≤ 7n/8.

Proof: We extend the result derived by A. Watanabe for p-solvable group. We prove here a generalized version of theorem (1.3) that “almost” yields bound for large primes. We show that it does not hold for an arbitrary group as derived by G.R. Robinson concerning Dade’s Conjecture.

Let F*(G) denote the generalized fitting subgroup of G. We know that for any group G, C_0(F*(G)) ≤ F*(G). It is applied for proving the three subgroups lemma when N = F*(G).

Theorem
Suppose that A acts faithfully and coprimely on a finite simple group G. Then A acts faithfully on any A-invariant Sylow p-subgroup of G.

Proof: D. Robinson derived that sporadic groups do not admit co-prime automorphism. By an appropriate application the classification of finite simple groups it implies that G is of Lie type.

Let p be the characteristic of G. We show that A acts faithfully on any A-invariant Sylow p-subgroup of G. Let us assume that A = [a] is cyclic of prime order. Let U be an A-invariant Sylow p-subgroup of G. We prove it contrary by assuming that [U, A] = 1, which implies that its rank is 1, let B = N_0(U).

Then U = F*(B), So that [B, A] = 1. If G has rank greater then 1, then G is generated by parabolic subgroups of the form N_0(V), where 1≤V ≤ U. We find that N_0(V) is A-invariant as [U, A] = 1. But for such Subgroups, F*(N_0(V)) = O_p(N_0(V)) ≤ U. So that a centralizes F*(N_0(V)) and it centralizes N_0(V). Thus G has a rank 1. We know that G is one of the groups, L_2(q), U_3(q), G_2(q), B_2(q) where q is a power of p (where q is an odd power of p in the last two cases, and p = 3, 2 respectively). In all cases, G is a doubly transitive group on cosets of B and we obtain [G:B]=1+|U|.

But G = B∪Bwhere w is an involution. Also w normalizes B∩B = T, and T is a Hall P-subgroup of B.

Hence G = B N_0(T). Hence [N_0(T) : T] = 2 in all cases, and as such [N_0(T), a] ≤ T and [N_0(T), a] ≤ [T, a] = 1. Thus [N_0(T), a] = 1 by co-prime action. So that [G, a] = 1 which is contrary to assumption of the theorem.

Theorem
Suppose that A acts coprimely as automorphism on a finite group G. If C_0(G) = 1, then there exists a nilpotent A-invariant subgroup H of G such that C_0(H) = 1.

Proof: We prove it by induction on |G|. Let Z = Z(G). Let us assume that Z = 1. If not possible, let D = C_0(Z). We thus find that A/D acts faithfully on G/Z. By induction there exists an A-invariant nilpotent Subgroup V of G/Z on which A/D acts faithfully. Hence V is A-invariant and nilpotent. Now, C_0(V)/D centralizes V/Z, and C_0(V) ≤ D. Since C_0(V) ≤ C_0(Z), it implies using co-prime action that C_0(V) ≤ C_0(G) = 1. We first show that G is not expressible as the product of two proper A-invariant normal subgroups G=GN with [M,N]=1. Let B = C_0(M), C = C_0(N) then B∩C = 1. Now A/B acts faithfully on M, hence by induction there is nilpotent A-invariant subgroup H of M with C_0(H) = B. Similarly, A/C acts faithfully on N. By induction there is a nilpotent A-invariant subgroup K of M with C_0(K) = C, then HK is nilpotent, A-invariant and C_0(HK)=1 we assume that G=F*(G) such that F*(G)=E(G)/F(G), Where E(G) is the layer of G and G=E(G). Now, by suitable application of the result due to T.R. Wolf, G = K_1 × K_2 ×⋯ × K_t, where K_i is the Simple non-abelian.

Now, A acts on {K_1, K_2,…, K_t}. Let, Δ_1, Δ_2, Δ_3 be the distinct A-arithds and Δ_1=Δ_2=Δ_3=Δ. If S>1, then A acts transitively on the set {K_1, K_2,…, K_t}. Let us write K_t = K^a for some a_i ∈ A, Where K=K_1 and a_i=1.B = N_0(K)/C_0(K).

We find that B acts faithfully on the non abelian simple group K. There exists a B-invariant Sylow q-subgroup 1 ≠ Q of K such that B acts faithfully on Q. We show that C_0(Q) ≤ C_0(K). Let a∈C_0(Q). Let us suppose that K^a = K_t for instance. Then Q = Q^a ≤ K_t∩K_t=1 and this does not hold. Therefore, a∈N_0(K). Hence, a(C_0(K) 2B centralizes Q and it is proved. Let Qi = Q^a and let U=Qi, which is again nilpotent. We thus find that C_0(Qi) ≤ C_0(Ki). Now, let a∈C_0(U). Hence, a ∈ C_0(Qi). Then a centralizes G and U is A-invariant. Let X∈A.

If K_1 = K_t is sufficient to show that Q^i = Qi. But K^a_t = K^a_t. Hence, axa^−1j∈N_0(K). Thus Q^a^−1j = Qi and Qi=Q_1, Hence the theorem is proved.

Reference