Observations on the negative Pellian \( y^2 = 86x^2 - 5 \)

S.Vidhyalakshmi, T.R.Usha Rani, M.A.Gopalan, V.Kumari

Abstract

The binary quadratic equation \( y^2 = 86x^2 - 5 \) is considered and a few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration, a special Pythagorean triangle is formed.

Keywords: Binary quadratic equation, Integral solutions. 
MSC subject classification: 11D09.

1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety..In [1-9] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions of an another interesting binary quadratic equation given by \( y^2 = 86x^2 - 5 \). The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

Method of Analysis

Consider the binary quadratic equation

\[ y^2 = 86x^2 - 5 \]  

(1)

with the least positive integer solutions \( x_0=1, y_0=9 \)

To obtain the other solutions of equation (1), consider the Pellian equation

\[ y^2 = 86x^2 + 1 \]

Whose general solution \((x_n, y_n)\) is given by \( x_n = \frac{g}{2\sqrt{86}}, \ y_n = \frac{f}{2} \) in which

\[ f = (10405 + 1122\sqrt{86})^{n+1} + (10405 - 1122\sqrt{86})^{n+1} \]
\[ g = (10405 + 1122\sqrt{86})^{n+1} - (10405 - 1122\sqrt{86})^{n+1} \]

Where \( n = -1, 0, 1, 2 \).........................

Applying Brahmagupta lemma between the solutions of \((x_0,y_0)\) and \((x_n, y_n)\) the general solutions of equation (1) are found to be

\[ x_{n+1} = \frac{f}{2} + \frac{9\sqrt{86}g}{172} \]  

(2)
Thus (2) and (3) represent non-zero distinct integer solutions of (1) which represents a hyperbola. The recurrence relations satisfied by the values of \( x \) and \( y \) are respectively

\[
x_{n+3} - 20810 x_{n+2} + x_{n+1} = 0
\]

\[
y_{n+3} - 20810 y_{n+2} + y_{n+1} = 0
\]

A few numerical examples are presented in the table below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{n+1} )</th>
<th>( y_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>20503</td>
<td>190137</td>
</tr>
<tr>
<td>1</td>
<td>42667429</td>
<td>3956750961</td>
</tr>
<tr>
<td>2</td>
<td>8878949176987</td>
<td>82339987308273</td>
</tr>
</tbody>
</table>

A few interesting properties are given below:

(i) \( y_{n+1} \equiv 0 \pmod{3} \)

(ii) \( x_{n+1} \) and \( y_{n+1} \) are always odd

(iii) \( x_{n+2} = 10405 x_{n+1} + 1122 y_{n+1} \)

(iv) \( 860 x_{n+3} = x_{n+1} (172 c_1 - 1548 c_2) + y_{n+1} (172 c_2 - 18 c_1) \)

Where \( c_1 = 1763258*10405 + 190137*86*1122 \)

\( c_2 = 1763258*1122 + 190137*10405 \)

(v) \( y_{n+2} = 96492 x_{n+1} + 10405 y_{n+1} \)

(vi) \( 10 y_{n+3} = x_{n+1} (172 D_2 - 1548 D_1) + y_{n+1} (172 D_2 - 18 D_3) \)

Where \( D_1 = 190137*10405 + 20503*86*1122 \)

\( D_2 = 190137*1122 + 20503*10405 \)

(vii) \( 30 (172 x_{2n+2} - 18 y_{2n+2} + 10) \) is a nasty number

(viii) \( 25 (172 x_{3n+3} - 18 y_{3n+3} + 15 (172 x_{n+1} - 18 y_{n+1})) \)

is a cubic integer

(ix) Define \( X = 172 x_{n+1} - 18 y_{n+1} \) and \( Y = 172 y_{n+1} - 1548 x_{n+1} \) then the pair \( (X,Y) \) satisfies the hyperbola \( Y^2 = 86 X^2 - 8600 \)

**Remarkable Observations**

Let \( m = (x+y) \), \( n = x \) be any two non-zero distinct positive integers. Note that \( m > n \). Treat \( m, n \) as the generators of the Pythagorean triangle \( T (\alpha, \beta, \gamma) \) where \( \alpha = 2mn, \beta = m^2 - n^2, \gamma = m^2 + n^2 \). Let \( A, P \) represent the area and perimeter of the Pythagorean triangle \( T \). Then the following relations are observed:

\[ a) \alpha + 42 \gamma - 43 \beta - 5 = 0 \]
\[ b) 44 \beta - 43 \gamma - \frac{4A}{P} + 5 = 0 \]
\[ c) 44 \alpha - \gamma - \frac{172A}{P} - 5 = 0 \]

**Conclusion**

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables

**References**

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