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On the convergence rate for a new family of linear positive operators

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Abstract
In 1967, Durrmeyer introduced the notion of summation-integral type operators to approximate Lebesgue integrable functions on [0, 1]. In the present paper, we introduced a new family of mixed summation integral type operators, to approximate Lebesgue integrable functions on the interval [0, ∞] and study the rate of convergence for these operators. The family of operators is constructed by combing the well known Beta and Baskakov operators. The estimate for rate of convergence is obtained in terms of higher order modulus of continuity, by using the techniques of linear combinations.

Keywords: Simultaneous approximation, Summation-integral type operators, Linear positive operators, Modulus of smoothness, Linear combination

1. Introduction

Let $C_\gamma[0,\infty)$, $\gamma \geq 0$ be the class of locally integrable functions f defined on $[0,\infty)$ and satisfying the growth condition $|f(t)| \leq K t^\gamma,\ t \in (0,\infty)$, for some $K > 0$. For $f \in C_\gamma[0,\infty)$ with norm $\|f\|_\gamma = \max\left\{ |f(0)|, \sup\limits_{0<t<\infty} |f(t)|\, t^{-\gamma} \right\}$, we introduce the following family of mixed summation integral type operators as

$$V_n(f,x) = \frac{n-1}{n}\sum_{v=1}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v-1}(t)\, f(t)\,dt + (1+x)^{-n-1} f(0),\ x \in [0,\infty), \qquad (1.1)$$

where $q_{n,v}(x) = \dfrac{1}{\beta(v+1,n)} x^v (1+x)^{-n-v-1}$, $\quad p_{n,v}(t) = \dbinom{n+v-1}{v} \dfrac{t^v}{(1+t)^{n+v}}$,

and $\beta(v+1,n)$ being the Beta function given by $v!(n-1)/(n+v)!$.

On the other hand, in terms of Dirac delta function $\delta(t)$, the operators (1.1) may be written as

$$V_n(f,x) = \int_0^\infty W_n(x,t) f(t)dt, \qquad 0 \leq x < \infty, \qquad (1.2)$$

Where the kernel is given by

$$W_n(x,t) = \frac{n-1}{n}\sum_{v=1}^{\infty} q_{n,v}(x) p_{n,v-1}(t) + (1+x)^{-n-1}\delta(t).$$

Since $f(t) = O(t^\gamma)$ for $f \in C_\gamma[0,\infty)$, and

$\int_0^\infty t^{r-1}/(1+t)^{s+m}\,dt$ exists for positive values of r, s only, it follows that for $f \in C_\gamma[0,\infty)$, the operators (1.1) are well defined for each $n > \gamma+1$.

We may observe that the operators defined by (1.1) are linear positive operators [9]. Also the operators V_n reproduce only the constant functions. It is easily verified that the order of approximation by these operators is at best $O\left(n^{-1}\right)$ howsoever smooth the function may be. For improving the order of approximation, we consider the linear combinations of the operators (1.1) as defined by

$$V_n(f,k,x) = \frac{1}{\Delta}\begin{vmatrix} V_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k} \\ V_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ V_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k} \end{vmatrix},$$

Where $d_0, d_1, d_2, \ldots, d_k$ are $(k+1)$ arbitrary but fixed distinct positive integers and Δ is the Vandermonde determinant obtained by replacing the operator column of the above determinant by the entries 1.

The above linear combinations in alternative form maybe rewritten as

$$V_n(f,k,x) = \sum_{j=0}^{k} C(j,k) V_{d_j n}(f,x) \tag{1.3}$$

Where $C(j,k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0, \; C(0,0) = 1.$

Such type of linear combinations were first considered by May [10] to get the better order of approximation for exponential type operators which include Bernstein polynomials, Baskakov operators, Szasz operators, Gauss Weierstrass operators etc. as special cases. Durrmeyer [4] introduced the notion of summation-integral type operators to approximate Lebesgue integrable functions on $[0,1]$. Several researchers proposed and studied rate of convergence for summation-integral type operators [1, 2, 3, 6, 7, 8, 11].

2. Preliminaries

To prove the main results, we need the following preparatory lemmas and proposition:

Lemma 2.1. For $m \in N^0$ and $x \in [0,\infty)$, if the function $\mu_{n,m}(x)$ be defined by

$$\mu_{n,m}(x) = \frac{n-1}{n} \sum_{v=1}^{\infty} q_{n,v}(x) \int_0^\infty p_{n,v-1}(t)(t-x)^m\,dt + (-x)^m (1+x)^{-n-1},$$

then, we have

$$\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \frac{3x}{n-2}, \quad \mu_{n,2}(x) = \frac{x(1+x)(2n-1) + 3x(1+5x)}{(n-2)(n-3)},$$

and there holds the following recurrence relation

$$(n-m-2)\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)] + [m(1+2x)+3x]\mu_{n,m}(x).$$

Furthermore, it follows from the above recurrence relation that

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right) \quad \text{as } n \to \infty,$$

Where [α] denotes the integral part of α.

Proof. The values of $\mu_{n,0}(x), \mu_{n,1}(x)$ easily follow from the definition.

Now, we have

$$x(1+x)\,\mu'_{n,v}(x)=\frac{n-1}{n}\sum_{v=1}^{\infty}x(1+x)\,q'_{n,v}(x)\int_0^{\infty}p_{n,v-1}(t)\,(t-x)^m\,dt$$
$$-m\frac{n-1}{n}\sum_{v=1}^{\infty}x(1+x)q_{n,v}(x)\int_0^{\infty}p_{n,v-1}(t)(t-x)^{m-1}dt$$
$$-[\,(n+1)(-x)^m(1+x)^{-n-2}+m(-x)^{m-1}(1+x)^{-n-1}]\,x(1+x)$$

Next using the well known identities

$$x(1+x)q'_{n,v}(x)=[v-(n+1)x]q_{n,v}(x) \qquad \text{and} \qquad t(1+t)\,p'_{n,v}(t)=(v-nt)\,p_{n,v}(t)\,,$$

we obtain

$$x(1+x)\big[\,\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\,\big]$$
$$=\frac{(n-1)}{n}\sum_{v=1}^{\infty}[v-(n+1)x]q_{n,v}(x)\int_0^{\infty}p_{n,v-1}(t)(t-x)^m\,dt+(n+1)(-x)^{m+1}(1+x)^{-n-1}$$
$$=\frac{(n-1)}{n}\sum_{v=1}^{\infty}q_{n,v}(x)\int_0^{\infty}\big[\{(v-1)-nt\}+n(t-x)+(1-x)\,\big]p_{n,v-1}(t)\,(t-x)^m\,dt+(n+1)\,(-x)^{m+1}(1+x)^{-n-1}$$
$$+\frac{(n-1)}{n}(1-x)\sum_{v=1}^{\infty}q_{n,v}(x)\int_0^{\infty}p_{n,v-1}(t)\,(t-x)^m\,dt+(n+1)\,(-x)^{m+1}(1+x)^{-n-1}$$
$$=\frac{(n-1)}{n}\sum_{v=1}^{\infty}q_{n,v}(x)\int_0^{\infty}\big[(1+2x)\,(t-x)+(t-x)^2+x(1+x)\,\big]p'_{n.v-1}(t)\,(t-x)^m\,dt$$
$$+n\mu_{n,m+1}(x)+\frac{(n-1)}{n}(1-x)\sum_{v=1}^{\infty}q_{n,v}(x)\int_0^{\infty}p_{n,v-1}(t)\,(t-x)^m\,dt+(-x)^{m+1}(1+x)^{-n-1}$$
$$=-[m(1+2x)+3x]\mu_{n,m}(x)+(n-m-2)\mu_{n,m+1}(x)-mx(1+x)\mu_{n,m-1}(x)\,.$$

This completes the proof of recurrence relation.

Corollary 2.2 [5]**.** Let γ and δ be two positive numbers, then for any $m>0$ there exists a constant C such that

$$\left\|\int_{|t-x|\geq\delta}W_n(x,t)t^{\gamma}\,dt\right\|_{C[a,b]}\leq C\,n^{-m}\,.$$

Proposition 2.3 [5]**.** For $m\in N$ and sufficiently large $n\in N$, there holds the following relation

$$V_n\big((t-x)^m,k,x\,\big)=n^{-(k+1)}\big[\,Q(m,k,x)+o(1)\,\big],$$

where $Q(m,k,x)$ are certain polynomials in x of degree at most m.

Lemma 2.4 [5]**.** For $0<a<a_1<a_2<b_2<b_1<b<\infty$ and sufficiently small $\delta>0$, let the (2k+2)th order Steklov mean $g_{2k+2,\delta}(t)$ corresponding to $g\in C_{\gamma}[0,\infty)$ be defined by

$$g_{2k+2,\delta}(t)=\delta^{-(2k+2)m}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}\cdots\int_{-\delta/2}^{\delta/2}[g(t)-\Delta_{\eta}^{2k+2}g(t)]\prod_{i=1}^{2k+2}dt_i$$

where $\eta = \dfrac{1}{2k+2}\sum_{i=1}^{2k+2} t_i$, $t \in [a,b]$ and $\Delta_{\eta}^{2k+2} g(t)$ is the (2k+2)th forward difference with step length η. Then we have

(i) $g_{2k+2,\delta}$ has continuous derivatives up to order $(2k+2)$ on $[a,b]$;

(ii) $\left\| g_{2k+2,\delta}^{(r)} \right\|_{C[a_1,b_1]} \le C_1 \delta^{-r} \omega_r(g,\delta,a_1,b_1)$, r = 1, 2, 3, …, (2k+2);

(iii) $\left\| g - g_{2k+2,\delta} \right\|_{C[a_2,b_2]} \le C_2\, \omega_{2k+2}(g,\delta,a_1,b_1)$;

(iv) $\left\| g_{2k+2,\delta} \right\|_{C[a_2,b_2]} \le C_3 \|g\|_{\gamma}$,

where $C1$, $C2$, $C3$ are certain unrelated constants independent of g and δ.

3. Main Results

In this section we state and prove the following direct results

Theorem 3.1. If (2k+2)th derivatives of $f \in C_{\gamma}[0,\infty)$ exist at a point $x \in (0,\infty)$, then we have

$$n^{k+1}\left[V_n(f,k,x) - f(x)\right] = \sum_{i=k+1}^{2k+2} \frac{f^{(i)}(x)}{i!} Q(i,k,x) + o(1) \qquad (3.1)$$

And

$$\lim_{n\to\infty} n^{k+1}\left[V_n(f,k+1,x) - f(x)\right] = 0 \qquad (3.2)$$

Where $Q(i,k,x)$ are certain polynomials in x of degree i.

Proof. By Taylor's finite expansion of $f(t)$ we have

$$f(t) = \sum_{i=0}^{2k+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{2k+2}, \text{ where } \varepsilon(t,x) \to 0 \text{ as } t \to x.$$

Now,

$$n^{k+1}\left[V_n(f,k,x) - f(x)\right]$$

$$= n^{k+1}\sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} V_n\left((t-x)^i, k, x\right) + n^{k+1}\sum_{j=0}^{k} C(j,k) V_{d_j n}\left(\varepsilon(t,x)(t-x)^{2k+2},\ x\right) = E1 + E2 \qquad \text{(say)}$$

Since

$$\mu_{d_j n, i}(x) = \frac{P_0(x)}{(d_j n)^{[(i+1)/2]}} + \frac{P_1(x)}{(d_j n)^{[(i+1)/2]+1}} + \dots + \frac{P_{[i/2]}(x)}{(d_j n)^{i}},$$

Where $P_i(x), i = 0, 1, 2, \dots, [i/2]$ are certain polynomials in x of degree at most $[i/2]$. Therefore, by Proposition 2.3, we get

$$\sum_{j=0}^{k} C(j,k)\mu_{d_j n, i}(x) = \frac{1}{\Delta}\begin{vmatrix} \dfrac{P_0(x)}{(d_0 n)^{[(i+1)/2]}} + \dfrac{P_1(x)}{(d_0 n)^{[(i+1)/2]+1}} + \dots + \dfrac{P_{[\nu/2]}(x)}{(d_0 n)^{i}} & d_0^{-1} & d_0^{-2} & \dots & d_0^{-k} \\ \dfrac{P_0(x)}{(d_1 n)^{[(i+1)/2]}} + \dfrac{P_1(x)}{(d_1 n)^{[(i+1)/2]+1}} + \dots + \dfrac{P_{[\nu/2]}(x)}{(d_1 n)^{i}} & d_1^{-1} & d_1^{-2} & \dots & d_1^{-k} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{P_0(x)}{(d_k n)^{[(i+1)/2]}} + \dfrac{P_1(x)}{(d_k n)^{[(i+1)/2]+1}} + \dots + \dfrac{P_{[\nu/2]}(x)}{(d_k n)^{i}} & d_k^{-1} & d_k^{-2} & \dots & d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\left[Q(i,k,x) + o(1)\right],\ i = (k+1), (k+2), \dots, (2k+2).$$

Where Δ is the Vandermonde determinant obtained by replacing the first column by the entries 1.
Consequently, we get

$$E_1 = n^{k+1}\sum_{i=1}^{2k+2}\frac{f(x)}{i!}\sum_{j=0}^{k}C(j,k)\mu_{d_j n,i}(x) = \sum_{i=k+1}^{2k+2}\frac{f^{(i)}(x)}{i!}Q(i,k,x)+o(1).$$

To prove the assertion (3.1), it is sufficient to show that $E_2 \to 0$ as $n\to\infty$.
For a given $\varepsilon>0$ there exists a $\delta>0$ such that for $0<|t-x|<\delta$, $|\varepsilon(t,x)|<\varepsilon$ and for $|t-x|\geq\delta$, $|\varepsilon(t,x)|(t-x)^{2k+2}\leq K t^{\gamma}$, for some positive constant *K*.
If $\phi_\delta(t)$ is the characteristic function of the interval $(x-\delta, x+\delta)$, then we have

$$|E_2| \leq n^{k+1}\sum_{j=0}^{k}C(j,k)\Big[V_{d_j n}\Big(|\varepsilon(t,x)|(t-x)^{2k+2}\phi_\delta(t),x\Big)+V_{d_j n}\Big(|\varepsilon(t,x)|(t-x)^{2k+2}[1-\phi_\delta(t)],x\Big)\Big] = E_{21}+E_{22}\,\text{(say)}$$

Applying Lemma 2.1 and Corollary 2.2, we obtain

$$|E_{21}| \leq \varepsilon\, n^{k+1}\sum_{j=0}^{k}|C(j,k)| \max_{0\leq j\leq k} \mu_{d_j n,2k+2}(x) \leq C_1\,\varepsilon$$

And

$$|E_{22}| \leq \varepsilon\, n^{k+1}\sum_{j=0}^{k}|C(j,k)| \int_{|t-x|\geq\delta} W_{d_j n}(x,t)|\varepsilon(t,x)|\ (t-x)^{2k+2}dt$$

$$\leq n^{k+1}\sum_{j=0}^{k}|C(j,k)| \int_{|t-x|\geq\delta} W_{d_j n}(x,t)\, t^{\gamma}dt \leq C_2\, n^{k+1}\sum_{j=0}^{k}|C(j,k)|n^{-m} = O(1)$$

For some $m\geq(k+1)$.
Finally, combining the estimates of *E*21 and *E*22, we get (3.1). The proof of (3.2) follows easily along the lines of the proof of (3.1), by noticing the fact that

$$V_n\Big((t-x)^i, k+1, x\Big) = O\Big(n^{-(k+2)}\Big).$$

This completes the proof.

Theorem 3.2. Let $f\in C_\gamma[0,\infty)$. If $f^{(p)}, 1\leq p\leq 2k+2$ exists and is continuous with the modulus of continuity $\omega\big(f^{(p)},\delta\big)$ on $[0,a]$. Then for sufficiently large $n\in N$, we have

$$\|V_n(f,k,\cdot)-f\|_{C[0,b]} \leq \max\left\{C_1 n^{-p/2}\omega(f^{(p)},n^{-1/2}), C_2 n^{-(k+1)}\right\}$$

Where $C_1 = C_1(k,p), C_2 = C_2(k,p,f)$ and $0<b<a.$

Proof. For all *t* > 0 and $x\in[0,a],$ we have

$$f(t) = \sum_{i=0}^{p}\frac{f^{(i)}(x)}{i!}(t-x)^i + \Big[f^{(p)}(\xi)-f^{(p)}(x)\Big]\frac{(t-x)^p}{p!} + h(t,x)\,\chi(t), \tag{3.3}$$

where ξ lies between *t* and *x* and $\chi(t)$ is the characteristic function of the set $(0,\infty)\setminus[0,a].$ The function $h(t,x)$ is bounded by $K t^{\gamma}|t-x|^p,\ K>0$.
Using (3.3), we have

$$V_n(f,k,x)=\sum_{j=0}^{k}C(j,k)V_{d_jn}(f,x)=\sum_{j=0}^{k}C(j,k)\int_0^{\infty}W_{d_jn}(x,t)f(t)dt$$
$$=\sum_{j=0}^{k}C(j,k)\int_0^{\infty}W_{d_jn}(x,t)\left[\sum_{i=0}^{p}\frac{f^{(i)}(x)}{i!}(t-x)^i+[f^{(p)}(\xi)-f^{(p)}(x)]\frac{(t-x)^p}{p!}+h(t,x)\chi(t)\right]dt$$
$$=\sum_{i=0}^{p}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{k}C(j,k)\int_0^{\infty}W_{d_jn}(x,t)(t-x)^i dt$$
$$+\sum_{j=0}^{k}C(j,k)\int_0^{\infty}W_{d_jn}(x,t)[f^{(p)}(\xi)-f^{(p)}(x)]\frac{(t-x)^p}{p!}dt+\sum_{j=0}^{k}C(j,k)\int_0^{\infty}W_{d_jn}(x,t)h(t,x)\chi(t)dt$$
$$=J_1+J_2+J_3 \qquad \text{(Say)}$$

Making use of Lemma 2.1, we get

$$J_1=f(x)+o\left(n^{-(k+1)}\right) \text{ Uniformly for all } x\in[0,a]$$

Since for all $\delta>0,$ we have

$$\left|f^{(p)}(\xi)-f^{p}(x)\right|\le\omega\left(f^{(p)},|\xi-x|\right)\le\omega\left(f^{(p)},|t-x|\right)\le\left[1+\frac{|t-x|}{\delta}\right]\omega\left(f^{(p)},\delta\right).$$

Therefore, we have

$$|J_2|\le\frac{\omega\left(f^{(p)},\delta\right)}{p!}\sum_{j=0}^{k}|C(j,k)|\int_0^{\infty}W_{d_jn}(x,t)\left[|t-x|^p+\frac{|t-x|^{p+1}}{\delta}\right]dt$$

Using Schwarz inequality and Lemma 3.1, we get

$$|J_2|\le\frac{\omega\left(f^{(p)},\delta\right)}{p!}\sum_{j=0}^{k}|C(j,k)|\left[O\left((d_jn)^{-p}\right)+\frac{1}{\delta}O\left((d_jn)^{-(p+1)}\right)\right]^{1/2}$$

Consequently, choosing $\delta=n^{-1/2}$, we obtain

$$|J_2|\le C_3n^{-p/2}\omega\left(f^{(p)},n^{-1/2}\right)$$

Next, choosing a positive number *is* such that $|t-x|\ge s$, we get

$$|J_3|\le\sum_{j=0}^{k}|C(j,k)|\int_{|t-x|\ge s}W_{d_jn}(x,t)t^{\gamma}|t-x|^p dt.$$

Applying Schwarz inequality and Corollary 2.2, we have

$$|J_3|\le\sum_{j=0}^{k}|C(j,k)|\left[\int_{|t-x|\ge s}W_{d_jn}(x,t)e^{2\gamma t}dt\right]^{1/2}\left[\int_{|t-x|\ge s}W_{d_jn}(x,t)(t-x)^{2p}dt\right]^{1/2}$$
$$\le\sum_{j=0}^{k}|C(j,k)|(Kn^{-m})^{-1/2}O\left((d_jn)^{-p/2}\right)\le C_4n^{-(k+1)}.$$

Finally, combining the estimates of *J*1, *J*2 and *J*3, the Theorem 3.2 follows.
This completes the proof.
Our next result is the rate of convergence in terms of higher order modulus of continuity, which is stated as

Theorem 3.3. Let $f \in C_\gamma[0,\infty)$ aid $0 < a_1 < a_2 < b_2 < b_1 < \infty$. Then for sufficiently large $n \in N$, we have

$$\| V_n(f,k,\cdot) - f \|_{C[a_2,b_2]} \le K\left\lfloor n^{-(k+1)} \| f \|_\gamma + \omega_{2k+2}\left(f, n^{-1/2}; a_1, b_1\right)\right\rfloor,$$

where the constant *K* is independent of f and n .

Proof. First by linearity property of the operators (1.3), we have

$$\| V_n(f,k,\cdot) - f \|_{C[a_2,b_2]} \le \left\| V_n\left((f - f_{2k+2,\delta}), k, \cdot\right)\right\|_{C[a_2,b_2]}$$

$$+ \left\| V_n\left(f_{2k+2,\delta}, k, \cdot\right) - f_{2k+2,\delta}\right\|_{C[a_2,b_2]} + \| f - f_{2k+2,\delta} \|_{C[a_2,b_2]} \qquad = A1 + A2 + A3 \text{ (say)}$$

By property (iii) of Lemma 2.4, we have

$$A_3 \le C_1\, \omega_{2k+2}\left(f^{(r)}, \delta; a_1, b_1\right)$$

Next by Theorem 3.1, we have

$$A_2 \le C_2 n^{-(k+1)} \sum_{j=k+1}^{2k+2} \| f_{2k+2,\delta}^{(j)} \|_{C[a_1,b_1]}$$

By using the interpolation property due to Goldberg and Meir [5] for each $j = k+1,\ k+2,\ \ldots,\ 2k+2,$ we have

$$\| f_{2k+2,\delta}^{(j)} \|_{C[a_1,b_1]} \le C_3\left[\| f_{2k+2,\delta} \|_{C[a_1,b_1]} + \| f_{2k+2,\delta}^{(2k+2)} \|_{C[a_1,b_1]}\right] \le C_4\left[\| f_{2k+2,\delta}^{(2k+2)} \|_{C[a_1,b_1]} + \| f \|_\gamma\right]$$

Again using Lemma 2.4, it follows that

$$\begin{aligned}
A_1 &= \left\| V_n\left((f - f_{2k+2,\delta}), k, \cdot\right)\right\|_{C[a_2,b_2]} \\
&\le \left\| \sum_{j=0}^{k} |C(j,k)| \int_0^\infty W_{d_j n}(x,t)\, | f(t) - f(x) | dt \right\|_{C[a_2,b_2]} \\
&\le \left\| \sum_{j=0}^{k} |C(j,k)| \left[\int_{|t-x|\le\delta} + \int_{|t-x|>\delta} \right] W_{d_j n}(t,x)\, | f(t) - f(x) | dt \right\|_{C[a_2,b_2]} \\
&\le \| f - f_{2k+2,\delta} \|_{C[a_2-\delta, b_2+\delta]} + C_5\, n^{-m} \| f \|_\gamma \\
&\le C_6\, \omega_{2k+2}\left(f, \delta;\ a_1, b_1\right) + C_5 n^{-m} \| f \|_\gamma
\end{aligned}$$

Finally, choosing $\delta = n^{-1/2}$, and combining the estimates of *A*1, *A*2 and *A*3, we get the desired result.
This completes the proof.

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