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Lebesgue Decomposition Theorem and its extension to Signed measures

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Abstract
By Lebesgue Decomposition Theorem for any σ −finite measurable space (X, $\mathcal{A}$, μ) we prove that if v be any other σ −finite measure on $\mathcal{A}$ then there exist a unique pair of measures (v_0, v_1) such that $v = v_0 + v_1$ and $v_0 \perp \mu$ and $v_1 \ll \mu$. The result can be extended to signed measures also.

Key words: Signed measure, Measurable space, Hahn-Decomposition, Orthogonal measures.

Introduction
Definition: Let $\mathcal{A}$ be any σ-algebra on the set X. ϑ be a set function defined on $\mathcal{A}$ s.t.
(1) $-\infty \leq \vartheta \leq \infty$ and ϑ takes at most one of the values -∞ and ∞.
(2) $\vartheta(\phi) = 0$
(3) ϑ is Countably additive, then ϑ is called a Signed Measure on the space (X, $\mathcal{A}$, ϑ).

***Example* 1**: Let μ be any measure and $\vartheta = -\mu$, then ϑ is a signed measure.
2. $Let\ \lambda\ and\ \mu$ be any two measures s.t. at least one of them is finite. Take $\vartheta = \lambda - \mu$. Then ϑ is a signed measure.

Properties:
(1) ϑ is finitly additive.
Proof: Let $E_1, E_2, \ldots, E_n$ be finitly many disjoint measurable sets. And E=$\bigcup_{i=1}^{n} Ei$, Define $E_i = \phi$ for $i > n$. Then (E_i) is a disjoint sequence of measurable sets,
Hence $\vartheta(\bigcup_{i=1}^{n} Ei) = \sum_{i=1}^{n} \vartheta(Ei)$.

(**2**) $If\ \vartheta(E) is$ finite and F is a measurable subset of E, then $\vartheta(F) is\ finite$.
Proof: E = F∪(E-F) $\Rightarrow \vartheta(E) = \vartheta(F) + \vartheta(E - F)$, Since $\vartheta(E)$ is finite it follows immediately that $\vartheta(F)$ is finite.

(**3**) ϑ is subtractive i.e. if A $\supset B$ be the measurable sets, and $\vartheta(B)$ is finite and then $\vartheta(A - B) = \vartheta(A) - \vartheta(B)$.
Proof: A= B∪(A-B) $\Rightarrow \vartheta(A) = \vartheta(B) + \vartheta(A - B) \Rightarrow \vartheta(A - B) = \vartheta(A) - \vartheta(B)$.

Definition: Let N be any measurable set such that $\vartheta(M)$ = 0 for every measurable subset M of N. Then N is called a null set of ϑ.

Remark:
(1) A measurable sub set of null set is a null set.
(2) A countable union of null sets is a null set.
Let {N_k} be any sequence of null sets for ϑ, write N = $\bigcup_{n=1}^{\infty} N_k$ and M be any measurable subset of N. Define $M_1 = N_1, M_2 = N_2 - N_1, M_3 = N_3 - (N_1 \cup N_2)$,…………

Then $\{M_k\}$ is a disjoint sequence of Null sets, and $\bigcup_{n=1}^{\infty} M_k = \bigcup_{n=1}^{\infty} N_k$ = N
M ⊂ N ⇒ $M = M \cap N$= M∩ $[\bigcup_{n=1}^{\infty} M_k] = \bigcup_{n=1}^{\infty} [M \cap M_k]$
$\Rightarrow \vartheta(M) = \sum_{k=1}^{\infty} \vartheta([M \cap M_k) = 0$. [As M_k is a null set]
Proved.

Lemma: Suppose $v \perp \mu$ and $v \ll \mu$ then $v = 0$.
Proof: Since $v \perp \mu$, we can find disjoint measurable sets A and B such that X= A ∪ B and $v = 0$ on A and $\mu = 0$ on B.
Let E be any measurable sub set of X, then E = E ∩ X = E ∩ (A ∪ B) = (E ∩ A) ∪ (E ∩ B)

Then $v(E) = v(E \cap A) + v(E \cap B) = v(E \cap B)$ [Because $v(E \cap A) = 0$]
As $\mu(E \cap B) = 0$ and $v \ll \mu$ we have $v(E \cap B) = 0$
$\Rightarrow v(E) = 0 \ \forall$ measurable sub set E of X.
$\Rightarrow v = 0$.

Note: Above said theorem can also be stated as that Hahn-Decomposition is unique up to null sets.

Definition: Let v be any signed measure and S be any measurable subset of X. if X-S is a null set for v. Then we say that v is supported on S.

Definition: Let v_1, v_2 be two signed measures, then there exist a measurable set S s.t. v_1 is supported on S and v_2 is supported on X-S . Then we say that v_1 orthogonal or singular to v_2 and we write as $v_1 \perp v_2$.
Remark: $v_1 \perp v_2 \Leftrightarrow v_2 \perp v_1$ therefore we can say that v_1 *and* v_2 are mutually orthogonal.

Lebesgue Decomposition Theorem: Let (X, $\mathcal{A}, \mu$) be any $\sigma-$finite measure space. Let v be any other $\sigma-$finite measure on $\mathcal{A}$ then there exist a unique pair of measures (v_0, v_1) such that $v = v_0 + v_1$ and $v_0 \perp \mu$ and $v_1 \ll \mu$.

Proof: Define $\lambda = \mu + v$, then λ is a σ-finite measure and it is obvious that $\mu \ll \lambda$ and $\ll \lambda$.
Hence by Radon Nikodym Theorem we can find a non-negative measurable functions f *and* g such that $\mu(E) = \int_E f \, d\lambda$ and $v(E) = \int_E g \, d\lambda$ for every measurable set E of X.
Define A = $\{ f = 0\}$ and B = $\{f > 0\}$ then A and B are disjoint and X = A ∪ B.
Since $f = 0$ *on* A we get $\int_A f \, d\lambda = 0 \Rightarrow \mu(A) = 0$.
Define v_0 (E) = v $(E \cap A)$ then v_0 is a measure and
$v_0(B) = v(B \cap A) = 0 \Rightarrow v_0(B) = 0$ [B∩ A = ϕ]
Shows that $v_0 \perp \mu$ ..……. (1)
Define v_1 (E) = v $(E \cap B)$ then v_1 is a measure and
$v(E) = v(E \cap X) = v\left(E \cap (A \cup B)\right)$
$= v(E \cap A) + v(E \cap B) = v_0(E) + v_1(E)$ for all measurable subsets E of X.
Shows that $v = v_0 + v_1$ ...…… (2)
To show $v_1 \ll \mu$
Suppose $\mu(E) = 0 \Rightarrow \int_E f \, d\lambda = 0 \Rightarrow f = 0$ a.e.[λ] on E,
Since $f > 0$ on E ∩ B
$\Rightarrow \lambda(E \cap B) = 0$ [Because $E \cap B$ is a null set]
$\Rightarrow v(E \cap B) = 0$ as $v \ll \lambda \Rightarrow v_1(E) = 0$
shows that $v_1 \ll \mu$ ………… (3)

Uniqueness: Suppose we also have $v = v_0' + v_1'$ where v_0' and v_1' are measures on $\mathcal{A}$ and $v_0' \perp \mu$, $v_1' \ll \mu$. As μ and v both are σ-finite there exist a disjoint sequence {Xn} of measurable sets s.t. X = $\bigcup_1^\infty X_n$
And $\mu(Xn) < \infty$, $v(Xn) < \infty$, ∀ n.
This implies that v_0, v_1, v_0', v_1' are all finite on Xn.
And on Xn we have $v = v_0 + v_1$, $v = v_0' + v_1'$
$\Rightarrow v_0 + v_1 = v_0' + v_1' \Rightarrow v_0 - v_0' = v_1' - v_1$ on Xn
$v_0 \perp \mu$, $v_0' \perp \mu \Rightarrow v_0 - v_0' \perp \mu$, on the other hand $v_1 \ll \mu$ and $v_1' \ll \mu$
therefore $v_1' - v_1 \ll \mu \Rightarrow v_0 - v_0' \ll \mu$. By the above lemma we obtain $v_0 - v_0' = 0$
$\Rightarrow v_0 = v_0'$ on Xn $\Rightarrow v_0 = v_0'$ on X also.
Similarly $v_1 = v_1'$ on X also. Hence Proved

Lebesgue Decomposition Theorem for signed measures: Let (X,$\mathcal{A}, \mu$) be any σ-finite measurable space, Let v be any finite signed measure, then there exist a unique pair (v_0, v_1) of finite signed measures s.t. $v = v_0 + v_1$, $v_0 \perp \mu$, $v_1 \ll \mu$.
Proof: Since v is finite it follows that v^+ is a finite measure
Define $\lambda = \mu + v^+$ then λ is a σ- finite measure and $\mu \ll \lambda$, $v^+ \ll \lambda$.
By Radon Nikodym Theorem we can find non-negative measurable functions f *and* g such that $\mu(E) = \int_E f \, d\lambda$ and $v^+(E) = \int_E g \, d\lambda$ for every measurable set E of X.
Define $A_1 = \{ f = 0\}$ and $B_1 = \{f > 0\}$ then A_1 and B_1 are disjoint measurable sets, X = $A_1 \cup B_1$ as $f = 0$ *on* A_1 we have $\int_{A_1} f \, d\lambda = 0 \Rightarrow \mu(A_1) = 0$
Let E be any measurable set such that $\mu(E) = 0$ which gives $\int_E f \, d\lambda = 0 \Rightarrow f = 0$ *a.e.*[λ] on E.
Since $f > 0$ on E∩ B_1, we get $\lambda(E \cap B_1) = 0$
$\Rightarrow \int_{E \cap B_1} g \, d\lambda = 0 \Rightarrow v^+(E \cap B_1) = 0$ [As $v^+ \ll \lambda$]
Thus we obtain disjoint measurable sets A_1 and B_1 such that $\mu(A_1) = 0$, $v^+(E \cap B_1) = 0$ whenever $\mu(E) = 0$,
X= $A_1 \cup B_1$ ……………..….(1)
In the same manner we can find measurable disjoint sets A_2 and B_2 such that X = $A_2 \cup B_2$,$\mu(A_2) = 0$,
$v^-(E \cap B_2) = 0$ whenever $\mu(E) = 0$ ……….. (2)
Define A= $A_1 \cup A_2$ and B = X- A then A and B are disjoint measurable sets and X = A∪ B
Define $v_0(E) = v(E \cap A)$, $v_1(E) = v(E \cap B)$ for any measurable set E.
As v is a finite signed measure it is clear that v_0, v_1 both are signed measures for any measurable sub set E of X
E = E∩ X = E∩ (A ∪ B) = (E ∩ A) ∪ (E ∩ B)
$\Rightarrow v(E) = v(E \cap A) + v(E \cap B)$
$= v_0(E) + v_1(E) \Rightarrow v = v_0 + v_1$
From A = $A_1 \cup A_2$ we get $\mu(A) \le \mu(A_1) + \mu(A_2)$
$\Rightarrow \mu(A) = 0$
Let E be any measurable sub set of B then
$v_0(E) = v(E \cap A) = v(\phi) = 0$
Shows that $v_0 \perp \mu$
To show that $v_1 \ll \mu$, assume that $\mu(E) = 0$, then from (1) and (2), we get
$v^+(E \cap B_1) = 0, \quad v^-(E \cap B_2) = 0$
$E \cap B = E \cap (A^C) = E \cap [(A_1 \cup A_2)^C] = E \cap [(A_1)^C \cap (A_2)^C]$
= E∩ $[B_1 \cap B_2]$
= (E∩ B_1) ∩ (E∩ B_2) [From (1) and (2)]
$\Rightarrow$ E∩ B ⊂ E∩ B_1, E∩ B ⊂ E∩ B_2
$\Rightarrow v^+$(E∩ B) = 0, v^-(E∩ B) = 0 [From (1) and (2)]
$\Rightarrow v(E \cap B) = 0 \Rightarrow v_1(E) = 0$
Shows that $v_1 \ll \mu$.

Uniqueness: Suppose we also have $v = v_0' + v_1'$ where v_0' and v_1' are finite signed measures on $\mathcal{A}$ and $v_0' \perp \mu$, $v_1' \ll \mu$. As μ and v both are σ-finite there exist a disjoint sequence {Xn} of measurable sets s.t. X = $\bigcup_1^\infty X_n$

and $\mu(Xn) < \infty, v(Xn) < \infty, \forall$ n.
This implies v_{0}, v_1, v_0', v_1' are all finite on Xn. And on Xn we have $v = v_0 + v_1, v = v_0' + v_1'$
$\Rightarrow v_0 + v_1 = v_0' + v_1' \Rightarrow v_0 - v_0' = v_1' - v_1$ on Xn
$v_0 \perp \mu,\ v_0' \perp \mu \Rightarrow v_0 - v_0' \perp \mu$, on the other hand $v_1 \ll \mu$ and $v_1' \ll \mu$. Therefore $v_1' - v_1 \ll \mu \Rightarrow v_0 - v_0' \ll \mu$
by the above lemma we obtain $v_0 - v_0' = 0$
$\Rightarrow v_0 = v_0'$ on Xn $\Rightarrow$ and $v_0 = v_0'$ X also.
Similarly $v_1 = v_1'$ on X. Hence Proved

References

1. Hewitt Edwin, Stromberg Karl. Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable, Graduate Texts in Mathematics 25, Berlin, Heidelberg, New York: Springer-Verlag,
   ISBN 978-0-387-90138-1,MR 0188387,Zbl 0137.03202
2. Halmos PR. Measure Theory, Springer, New York, 1974.
3. Kumar A, Rao J.V. Jordan Decomposition and its Uniqueness of Signed Lattice Measure. Int J Contemp Math Sciences. 2011; (6)9: 431–438.
4. Qiang G. Further discussion on the Hann Decomposition Theorem for signed fuzzy measure, Fuzzy Sets and Systems. 1995; 70:89-95.
5. Walter Rudin. Real and Complex Analysis, McGraw-Hill, New York, 1966.American Mathematical Monthly. 1971; 78:7.