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A study on stability results for stochastic nonlinear systems of difference equations

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Abstract
In this paper, we present some results for the asymptotic stability of solutions for nonlinear fractional difference equations involving Riemann-Liouville-like difference operator. The results are obtained by using Krasnoselskii's fixed point theorem and discrete Arzela-Ascoli's theorem. Three examples are also provided to illustrate our main results.

Keywords: Nonlinear fractional difference, asymptotic stability solution, Riemann-Liouville-like difference operator.

Introduction
We consider the asymptotic stability of solutions for nonlinear fractional difference equations:

$$\Delta^{\alpha}x(t) = f(t+\alpha, x(t+\alpha)), t \in N_0, 0 < \alpha \leq 1,$$

$$\Delta^{\alpha-1}x(t)|_{t=0} = x_{0,}$$

where Δ^{α} is a Riemann-Liouville-like discrete fractional difference, $f: (0, +\infty)$ x $R \to R$ is continuous with respect to t and x, $N_a = \{a, a+1, a+2, \dots\}$.
Fractional differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering [1]. Most of the present works were focused on fractional differential equations, and the references there in. However, very little progress has been made to develop the theory of the analogous fractional finite difference equation [3].
Due to the lack of geometry interpretation of the fractional derivatives, it is difficult to find a valid tool to analyze the stability of fractional difference equations [5]. In the case that it is difficult to employ Liapunov's direct method, fixed point theorems are usually considered in stability. Motivated by this idea, in this paper, we discuss asymptotic stability of nonlinear fractional difference equations by using Krasnoselskii's fixed point theorem and discrete Arzela-Ascoli's theorem. Examples are provided to illustrate the main results.
We introduce preliminary facts of discrete fractional calculus.

Preliminaries
Definition 2.1 Let ν > 0. The ν-th fractional sum x is defined by

$$\Delta^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{(\nu-1)}f(s), \tag{2.1}$$

Where f is defined for $s = a$ mod (1) and $\Delta^{-\nu}f$ is defined for $t = (a+\nu)$ mod (1), and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$ The fractional sum $\Delta^{-\nu}$ maps functions defined on N_a to functions defined on $N_{a+\nu}$

Definition 2.2 Let $\mu > 0$ and $m - 1 < \mu < m$, where m denotes a positive integer, $m = [\mu]$ ceiling of number. Set $v = m - \mu$. The μ-th fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-v} f(t) = \Delta^{m}(\Delta^{-v} f(t)) \qquad (2.2)$$

Let f be real –value function defined on N_a and $\mu, v > 0$, then the following equalities hold

$$(i)\Delta^{-v}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+v)} f(t) = \Delta^{-\mu}[\Delta^{-v} f(t)];$$

$$(ii)\Delta^{-v}\Delta f(t) = \Delta\Delta^{-v} f(t) - \frac{(t-a)^{(v-1)}}{\Gamma(v)} f(a).$$

Stochastic Non-Linear System

We say that $S_{\mu}, r > 0$ is stable in probability if for any $c > r$ and any $\epsilon \in (0,1)$, there exist $\delta = \delta(\epsilon, c)$ such that for all $t \geq t_0 \geq 0$ and $|x_0| < \delta$, the following conditions holds:

$$P\{|x(t)| < c\} \geq 1 - \epsilon \qquad (3.1)$$

In the following we consider the Stochastic Non-linear System (**SNS**)

$$dx = (f(t,x)dt + h(t,x)u) + g(t,x)dw_t, \qquad (3.2)$$

Where $x \in \mathbb{R}^n, u \in \mathbb{R}^p$, the dynamics $f(t,x), g(t,x)$ and $h: \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times p}$, are continuous and Lipschitz functions with, $f(t,0) \neq 0$ and $g(t,0) \neq 0$.

This functions extension of 3.2.1 provided in that guarantees the existence of a C^{∞} control law $u = \emptyset(t,x)$ in such a way that S_r, satisfy the stability in

$$dx = \left(f(t,x)dt + h(t,x)\emptyset(t,x)\right) + g(t,x)dw_t$$

Denote by **D** the infinitesimal generator of the stochastic process solution of the uncontrolled part of **SNS** equation 3.2, that is, **D** is the second order differential operator defined for any function Φ *in* $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n, R)$ by

$$D_Z\Phi(t,x) = \frac{\partial\Phi(t,x)}{\partial t} + \sum_{i=1}^{n} f^i \frac{\partial\Phi(t,x)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} g_k^i g_k^j \frac{\partial^2\Phi(t,x)}{\partial x_i \partial x_j}$$

For any $z\epsilon(1,\dots,p)$, denote by D_z the first order differential operator defined for any function $\emptyset$ *in* $C^{1,1}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$ by

$$D_z\Phi(t,x) = \sum_{i=1}^{n} h_z^i(t,x)\frac{\partial\Phi(t,x)}{\partial x_i}$$

Define **X** as the infinitesimal generator for the stochastic process solution of the closed-loop system 3.3, that is, **X** is the differential operator defined for any function for any function Φ in $C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$ by

$$X\Phi(t,x) = D\Phi(t,x) + \sum_{z=1}^{p} D_z\Phi(t,x)u_z$$

In the following we extend the concept of stochastic Lyapunov condition introduced in definition provided used for stability in probability of **SNS** equation 3.2 at the neighborhood of the origin.

Application

In this section we illustrate our results by a designing a numerical example.

First Example

Consider the Stochastic Non-Linear System

$$d\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1 + x_2^3\\2\end{pmatrix}dt + \begin{pmatrix}0\\1\end{pmatrix}udt + \begin{pmatrix}x_1\\\frac{1}{2}\end{pmatrix}dw_t \qquad (4.1)$$

Where $(w_{t\geq 0})$ a standard real-valued Wiener process, u is a real-valued measurable control law,

$$f_1(t,x_1,x_2) = x_1 + x_2^3, f_2(t,x_1,x_2) = 2,$$

$$h_1(t,x_1,x_2) = 0, h_2(t,x_1,x_2) = 1,$$

$$g_1(t,x_1,x_2) = x_1 \; and \; g_2 = \; (t,x_1,x_2) = \frac{1}{2}$$

Define the Lyapunov function in the form

$$\Phi(t,x_1,x_2) = 2x_1^2\exp(2t) + (x_2 + x_1\exp(t))^4$$

A simple calculation shows that

$$\frac{\partial\Phi(t,x_1,x_2)}{\partial x_2} = 0 \Leftrightarrow x_2 = -x_1\exp(t) \qquad (4.2)$$

Therefore,

$$X\Phi(t,x_1,x_2) = D\Phi(t,x) + \sum_{z=1}^{p} D_z\Phi(t,x)u_z$$

$$= 10x_1^2\exp(2t) - 4x_1^4\exp(5t)$$

$$= 5\Phi(t,x_1,x_2) - \exp(t)\Phi^2(t,x_1,x_2)$$

$$\leq -\frac{1}{2}\Phi^2(t,x_1,x_2) + 5\exp(-t)$$

The later inequality implies that are fulfilled with $(s) = \frac{1}{2}s^2, \mu(t) = 5\exp(-t)$, $a_1(s) = \frac{1}{2}s^2$ and $a_2(s) = 6s^2$, and by theorem there exist a C^{∞} feedback law $\Phi(t,x_1,x_2)$ with $\Phi(t,0,0)$ such that S_r is stable in probability with respect to the resulting closed-loop system deduced from 4.1.

Homogeneous Difference Equations

When the number of lags grows large (3 or greater), solving linear difference equations by substitution is tedious. The key to understanding linear difference equations is the study of the homogeneous portion of the equation [8]. In the general linear difference equation,

$$y_t = \emptyset_0 + \emptyset_1 y_{t-1} + \emptyset_2 y_{t-2} + \cdots + \emptyset_p y_{t-p} + x_t$$

The homogenous portion is defined as the terms involving only y,

$$y_t = \emptyset_1 y_{t-1} + \emptyset_2 y_{t-2} + \cdots + \emptyset_p y_{t-p} \qquad (5.1)$$

The intuition behind studying this portion of the system is that, given the sequence of $\{x_t\}$, all of the dynamics and the stability of the system are determined by the determined by the relationship between contemporaneous y_t and it's lagged values which allows the determination of the parameter values where the system is stable [15]. Again, consider the homogeneous portions of the simple 1st order system

$$y_t = \emptyset_1 y_{t-1} + x_t$$

Which has homogenous portion

$$y_t = \emptyset_1 y_{t-1}$$

It is easy to show that

$$y_t - \emptyset_1^t y_0$$

Is also a solution by examining the solution to the linear difference equation. The solution of the form $c\emptyset_1^t$ for an arbitrary constant c.

$$y_t = c\emptyset_1^t$$

$$y_{t-1} = c\emptyset_1^{t-1}$$

and

$$y_t = \emptyset_1 y_{t-1}$$

Putting these two together shows that

$$y_t = \emptyset_1 y_{t-1}$$

$$c\emptyset_1^t = \emptyset_1 y_{t-1}$$

$$c\emptyset_1^t = \emptyset_1 c\emptyset_1^{t-1}$$

$$c\emptyset_1^t = c\emptyset_1^t$$

Linear Homogeneous Difference Equations

An equation of the form

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = k,$$

in which a_1 and a_2 are constants, is a linear second-order difference equation, with constant coefficients. This is precisely the type of equation we found for Y_t in the previous section. When k=0, we have the homogeneous equation

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = 0$$

It follows that if we know two solutions $y_t^{(1)}$ and $y_t^{(2)}$ of the difference equation, then

$Ay_t^{(1)} + By_t^{(2)}$ is also a solution for any constants A and B. Suppose we are given a homogeneous difference equation $y_t + a_1 y_{t-1} + a_2 y_{t-2} = 0$. In order to determine the sequence of values y_t completely we must know the initial values y_0 and y_1. Given these values, y_2 is determined by the equation with t=2, y_3 is then determined by the equation with t=3 and so on. So if we are looking for a solution $y_t = Ay_t^{(1)} + By_t^{(2)}$, we have to choose A and B so that the formula fits the initial conditions when t=0 and t=1. These two conditions determine appropriate values for the two arbitrary constants [12]. This means that the general solution of the homogeneous difference equation is given by the formula displayed above.

We shall now describe a practical method for finding two solutions $y_t^{(1)}$ and $y_t^{(2)}$, based on the auxiliary equation

$$z^2 + a_1 z + a_2 = 0$$

This is a quadratic equation. We observed that such an equation may have two distinct solutions, or just one solution, or no solutions, depending on the value of the quantity $a_1^2 - 4a_2$.

The auxiliary equation has just one solution with another example

It is clear that we cannot get a solution involving two arbitrary constants by the method used above. If the (one) solution of the auxiliary equation is α, then $y_t = \alpha^t$ is a solution of the difference equation as before, but we need to find another.

The auxiliary equation has exactly one solution when $a_1^2 - 4a_2 = 0$; that is, when $a_2 = \frac{a_1^2}{4}$. Then the equation $z^2 + a_1 z + a_2 = 0$ can be written in the form

$$z^2 + a_1 z + \frac{a_1^2}{4} = 0, \quad or \quad \left(z + \frac{a_1}{2}\right) = 0,$$

and the (one) solution is $\alpha = -\frac{a_1}{2}$. We claim that a second solution of the difference equation is $y_t = t\alpha^t$. Substituting this,

$$y_t + a_1 y_{t-1} + a_2 y_{t-2} = t\alpha^t + a_1(t-1)\alpha^{t-1} + a_2(t-2)\alpha^{t-2}$$
$$= t\alpha^{t-2}(\alpha^2 + a_1\alpha + a_2) - \alpha^{t-2}(a_1\alpha + 2a_2)$$

Because α satisfies the auxiliary equation, we have $\alpha^2 + a_1\alpha + a_2 = 0$. Furthermore, since $\alpha = -\frac{a_1}{2}$ and $a_1^2 = 4a_2$, it follows that $a_1\alpha + 2a_2 = -\frac{a_1^2}{2} + 2a_2 = 0$. Hence $t\alpha^t$ is a solution, as claimed.

The general solution is therefore

$$y_t = Ct\alpha^t + Dt\alpha^t = (Ct + D)\alpha^t$$

The values of the constants C and D can be determined by using the initial values y_0 and y_1.

Third Example: Consider the difference equation

$$y_0 = 1, y_1 = 1, y_t - 6y_{t-1} + 9y_{t-2} = 0$$

The auxiliary equation is

$$z^2 - 6z + 9 = 0, that\ is\ (z-3)^2 = 0$$

There is therefore just one solution, $\alpha = 3$, of the auxiliary equation. The general solution to the difference equation is (Ct+D)3^t [17]. Using the facts that $y_0 = 1$ and $y_1 = 1$, we must have

D=1, 3(C+D)=1

So that C=-2/3 and D=1, giving

$$y_t = \left(-\frac{2}{3}t + 1\right)3^t$$

Conclusion

In This paper we derived Stability in probability of stochastic nonlinear system there are many types of stochastic system although they do not trivial solution. We have established the stability probability of non-trivial solutions for stochastic nonlinear system. We have derived a stochastic version of control Lyapunov function and provided the necessary and sufficient condition in probability of a non-trivial solution for stochastic non-linear system exits. The numerical examples are solved to illustrate our results

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