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M Navaneethakrishnan
Department of Mathematics,
Kamaraj College,
Thoothukudi, Tamilnadu,
India.

P Periyasamy
Department of Mathematics,
Kamaraj College,
Thoothukudi, Tamilnadu,
India.

S Pious Missier
Department of Mathematics,
V.O. Chidambaram College,
Thoothukudi, Tamilnadu,
India.

Correspondence
P Periyasamy
Assistant Professor,
Department of Mathematics,
Kamaraj College, hoothukudi,
Tamilnadu, India.

$\hat{\delta}_r$-closed sets in Ideal Topological Spaces

M Navaneethakrishnan, P Periyasamy, S Pious Missier

Abstract

In this paper we introduce and study a new class of sets called $\hat{\delta}_r$ –closed sets in ideal topological spaces. Also we investigate the relationships with some of the known closed sets. This new class of sets is a collection of subsets of X which is independent of closed, semi-closed, α-closed, g-closed and Ig-closed sets.

Keywords: $\hat{\delta}_r$ –closed sets, $\hat{\delta}_r$ –open sets

1. Introduction

The concept of generalized closed sets and generalized open sets were first introduced by N. Levine [10] in usual topological spaces and regularopen sets have been introduced and investigated by Stone [20]. The concept of regular generalized closed sets in a topological space were studied by N. Palaniappan [19]. Levine [11], Njastad [18], Velicko [22], Bhattacharya and Lahiri [5], Arya and Nour [4], Maki, Devi and Balachandran [12], Dontchev and Maki [7], Dontchev and Ganster [6], introduced and investigated, semi-open sets, α-open sets, θ-closed sets, δ-closed sets, sg-closed sets, gs-closed sets, gα-closed sets, αg-closed sets, θ-g-closed sets, δg-closed sets respectively. The purpose of this paper is to introduce and study the notion of $\hat{\delta}_r$ –closed sets. Also we investigate the relationship with some known closed sets and verify this class of sets is independent of closed sets, semi-closed sets, α-closed sets, g-closed sets and Ig-closed sets.

2. Preliminaries

Definition 2.1 A subset A of a topological space (X, τ) is called

(i) regularopen set [20] if A = int cl(A)
(ii) Semi-open set [11] if $A \subseteq cl(int(A))$
(iii) Pre-open set [14] if $A \subseteq int(cl(A))$
(iv) α-open set [18] if $A \subseteq int(cl(int(A)))$
(v) θ-closed set [22] if $A = cl_\theta(A)$, where $cl_\theta(A) = \{x \in X : cl(U) \cap A \neq \phi$, for each $U \in \tau$ and $x \in U\}$.
(vi) δ-closed set [22] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x \in X: int(cl(U)) \cap A \neq \phi$, for each $U \in \tau$ and $x \in U\}$.

The complement of regularopen (resp. semi-open, pre-open, α-open) sets are called regularclosed (resp. semi-closed, pre-closed, α-closed). The complement of θ-closed (resp. δ-closed) sets are called θ-open (resp. δ-open). The semi-closure (resp. pre-closure, α-closure, R-closure) of a subset A of (X, τ) is the intersection of all semi-closed (resp. pre-closed, α-closed, regularclosed) sets containing A and is denoted by scl(A) (resp. Pcl(A), αcl(A), Rcl(A)). The intersection of all open (resp. α-open, semi-open, regularopen) sets of (X, τ) containing A is called kernel (resp. α-kernel, semi-kernel, regular-kernel) of A and is denoted by ker (resp. (αker, sker)(A), $A_r^{\wedge}$).

Definition 2.2 Let (X, τ) be a topological space. A subset A of X is said to be

(i) g-closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ)

(ii) Generalized semi-closed (briefly gs-closed) set [4] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ).

(iii) Semi-generalized closed (briefly sg-closed) set [5] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open in (X, τ).

(iv) a α-generalized closed (briefly αg-closed) set [12] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(v) a generalized α-closed (briefly gα-closed) set [13] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ).

(vi) δ-generalized closed (briefly δg-closed) set [8] if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(vii) $\hat{g}$ (or) w-closed set [21] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open set in (X, τ). The complement of $\hat{g}$ (or) w-closed set is $\hat{g}$ (or) w-open.

(viii) $\alpha\hat{g}$-closed set [1] if $\alpha cl(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}$-open set in (X, τ).

(ix) $\delta\hat{g}$-closed set [9] if $cl_\delta(A) \subseteq U$, whenever $A \subseteq U$ and U is $\hat{g}$-open set in (X, τ).

(x) θ-g-closed set [7] if $A = cl_\theta(A)$.

Definition 2.3 Let (X, τ, I) be an ideal space. A subset A of X is said to be

(i) δ-I-closed set [23] if $\sigma cl(A)=A$ where $\sigma cl(A) = \{x\in X: int(cl^*(U)) \cap A \neq \phi$, for each open set U and $x\in U\}$.

(ii) $\hat{\delta}$-closed set [16] if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ, I).

(iii) $\hat{\delta}_s$-closed set [17] if $\sigma cl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open in (X, τ, I).

(iv) Ig-closed set [6] if $A^* \subseteq U$, whenever $A \subseteq U$ and U is open in (X, τ, I).

(v) θ-I-closed [2] if $cl^*_\theta(A) = A$, where $cl^*_\theta(A) = \{x\in X: cl^*(U) \cap A \neq \phi$ for all $U\in\tau$ and $x\in U\}$.

3. $\hat{\delta}_r$-closed sets

In this section we introduce and study a new generalized closed sets called $\hat{\delta}_r$-closed sets.

Definition 3.1 A subset A of an ideal space (X, τ, I) is said to be $\hat{\delta}_r$-closed if $\sigma cl(A) \subset U$, whenever $A \subset U$ and U is regularopen. The complement of $\hat{\delta}_r$-closed set is called $\hat{\delta}_r$-open set.

Theorem 3.2 Every $\hat{\delta}$ (resp. $\hat{\delta}_s$) – closed set is $\hat{\delta}_r$-closed.

Proof. The proof is follows from the definition, and the fact that every regularopen set is open and semi-open.

Remark 3.3 The following Example shows that the reversible implication is not always holds.

Example 3.4 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{b, d\}$. Then A is $\hat{\delta}_r$-closed set but not $\hat{\delta}$-closed and not $\hat{\delta}_s$-closed.

Theorem 3.5 Every δ (resp. δ-I, δg, $\delta\hat{g}$, θ, θ-g, θ-I)-closed set is $\hat{\delta}_r$-closed.

Proof. By [16], every δ (resp. δ-I, δg, $\delta\hat{g}$, θ, θ-g, θ-I)-closed set is $\hat{\delta}$-closed, and by Theorem 3.2, it holds.

Remark 3.6 The reversible direction of Theorem 3.5, is not always true as shown in the following example.

Example 3.7 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ and $I = \{\phi, \{c\}\}$.

(i) $\hat{\delta}_r$-closed: {X, ϕ, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}.

(ii) δ-closed sets: {X, ϕ, {c, d}, {a, c, d}, {b, c, d}}.

(iii) δ-I-closed sets: {X, ϕ, {c, d}, {a, c, d}, {b, c, d}}.

(iv) θ-closed sets: {X, ϕ}.

(v) θ-I-closed sets: {X, ϕ}.

(vi) θ-g-closed sets: {X, ϕ, {c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

(vii) δg-closed sets: {X, ϕ, {c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

(viii) $\delta\hat{g}$-closed sets: {X, ϕ, {c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

Remark 3.8 The following Examples shows that $\hat{\delta}_r$-closed sets are independent of closed (resp. semi-closed, sg-closed, gs-closed, Ig-closed α-closed, g-closed).

Example 3.9 In Example 3.7,

(i) Sg-closed sets: {X, ϕ, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, c, d}, {b, c, d}}.

(ii) Gs-closed sets: {X, ϕ, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

(iii) α-closed: {X, ϕ, {c}, {d}, {c, d}, {a, c, d}, {b, c, d}}.

(iv) G-closed: {X, ϕ, {c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

(v) Ig-closed sets: {X, ϕ, {c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}.

Example 3.10 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi, \{b\}\}$. Let $A = \{b\}$. Then A is Ig-closed set but not $\hat{\delta}_r$-closed.

Example 3.11 Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \phi, \{c\}, \{e\}, \{c, e\}, \{a, b, c\}, \{c, d, e\}, \{a, b, c, e\}\}$ and $I = \{\phi\}$. Let $A = \{a, b\}$. Then A is closed, α-closed, g-closed. But A is not $\hat{\delta}_r$-closed.

4. Characterization

Theorem 4.1 Let (X, τ, I) be an ideal topological space. Then $\sigma cl(A)$ is always $\hat{\delta}_r$-closed for every subset A of X.

Proof. Let $\sigma cl(A) \subseteq U$, where U is regularopen set. Since $\sigma cl(\sigma cl(A)) = \sigma cl(A)$, $\sigma cl(A)$ is $\hat{\delta}_r$-closed.

Theorem 4.2 If A is $\hat{\delta}_r$-closed subset of an ideal space (X, τ, I), then σcl(A) – A does not contain any non-empty regularclosed set in (X, τ, I).

Proof. Let F be any regularclosed set in (X, τ, I) such that F $\subseteq$ σcl(A) – A, then A $\subset$ X–F. Since X–F is regularopen set containing A, by hypothesis σcl (A) $\subseteq$ X–F. Hence F $\subseteq$ X–σcl (A). Therefore F $\subseteq$(σcl(A)–A)$\cap$(X–σcl(A)) = ϕ.

Remark 4.3 The converse of Theorem 4.2 is not always true as shown in the following Example.

Example 4.4 In Example 3.11, let A = {a, b}, then σcl(A)–A = {a, b, c, d} – {a, b} = {c, d}, does not contain any non empty regularclosed set. But A is not $\hat{\delta}_r$-closed.

Theorem 4.5 If A is $\hat{\delta}_r$-closed subset of (X, τ, I), then σcl(A)–A does not contain any non- empty semi-open pre-closed subset of X.

Proof. The proof is follows from the fact that every semi-open pre-closed subset of X is regularclosed and by Theorem 4.2.

Remark 4.6 The converse of Theorem 4.5 is not true as shown in the following Example.

Example 4.7 In Example 3.7, let A= {a}. Then σcl (A)–A = {a, c, d} – {a} = {c, d} does not contain any non-empty semi-open pre-closed set. But {a} is not $\hat{\delta}_r$-closed.

Theorem 4.8 If A is both semi-open and pre-closed set in an ideal space (X, τ, I), then A is $\hat{\delta}_r$-closed in (X, τ, I).

Proof. It is clear that if A is both semi-open and pre-closed, then A is regularclosed and hence it is δ-closed in (X, τ, I). Therefore A is $\hat{\delta}_r$-closed in (X, τ, I).

Remark 4.9 The following Example shows that the converse of Theorem 4.8 is not true.

Example 4.10 In Example 3.7, let A = {c, d}, then A is $\hat{\delta}_r$-closed. But A is pre-closed not semi-open.

Theorem 4.11 If A is both regularopen and $\hat{\delta}_r$-closed subset of (X, τ, I). Then A is δ-I-closed subset of (X, τ, I).

Proof. Since A is regularopen and $\hat{\delta}_r$-closed, σcl(A) $\subseteq$ A. Therefore A is δ-I-closed.

Remark 4.12 The following Example shows that the converse of Theorem 4.11 is not true.

Example 4.13 In Example 3.7, let A = {a, c, d}. Then A is δ-I-closed. But A is $\hat{\delta}_r$-closed not regularopen.

Theorem 4.14 Let A be any $\hat{\delta}_r$-closed set in (X, τ, I). Then A is δ-I-closed in (X, τ, I) if and only if σcl(A)–A is regularclosed set in (X, τ, I).

Proof. Necessity - Since A is δ-I-closed set in (X, τ, I), σcl(A)–A = ϕ. Therefore σcl(A)–A is regularclosed in (X, τ, I).

Sufficiency. Since A is $\hat{\delta}_r$-closed set in (X, τ, I), By Theorem 4.2, σcl(A)–A does not contain any non-empty regularclosed set. Therefore, σcl(A)–A = ϕ. Hence A is δ-I-closed.

Theorem 4.15 Let (X, τ, I) be an ideal space. Then every subset of X is $\hat{\delta}_r$-closed if and only if every regularopen subset of X is δ-I-closed.

Proof. Necessity - Suppose every subset of X is $\hat{\delta}_r$-closed. If U is regularopen subset of X, then U is $\hat{\delta}_r$-closed and so σcl(U) = U. Therefore U is δ-I-closed.

Sufficiency - Suppose A $\subseteq$ U and U is regular open. By hypothesis U is δ-I-closed. Therefore σcl(A) $\subset$ σcl(U) = U. Therefore A is $\hat{\delta}_r$-closed.

Corollary 4.16 Let (X, τ, I) be an ideal space. Then every subset of X is δg-closed then every regularopen subset of X is δ-I-closed.

Theorem 4.17 A subset A of an ideal space (X, τ, I) is $\hat{\delta}_r$-closed if and only if σcl(A) $\subset$ $A_r^{\wedge}$.

Proof. Necessity - Suppose A is $\hat{\delta}_r$-closed and x$\in$σcl(A). If x$\notin$ $A_r^{\wedge}$, then there exists a regularopen set U containing A, but not containing x. Since A is $\hat{\delta}_r$-closed, σcl(A) $\subset$ U and so x$\notin$σcl(A), a contradiction. Therefore σcl(A) $\subset$ $A_r^{\wedge}$.

Sufficiency - Suppose that σcl(A) $\subset$ $A_r^{\wedge}$. If A$\subset$U and U is regularopen then $A_r^{\wedge}$ $\subset$ U and therefore σcl(A) $\subset$ U. Therefore A is $\hat{\delta}_r$-closed.

Corollary 4.18 If A is a δg-closed subset of an ideal space (X, τ, I), then σcl(A) $\subset$ $A_r^{\wedge}$.

Remark 4.19 The converse of the corollary 4.18 is not always true as shown in the following Example.

Example 4.20 In Example 3.4, let A = {a, b}. Then σcl({a, b}) = X $\subset$ X = $A_r^{\wedge}$. But A is not $\hat{\delta}_r$-closed set.

Definition 4.21 [15] Let (X, τ) be a topological space and A$\subset$X, then

(i) define $A_r^{\vee}$ = $\cup$ {F : F$\subset$A and F is regularclosed}.

(ii) A is said to be a $\vee_r$ – set if A = $A_r^{\vee}$

(iii) A is said to be a $\wedge_r$ – set if A = $A_r^{\wedge}$

Theorem 4.22 Let A be a $\wedge_r$-set of an ideal space (X, τ, I). Then A is $\hat{\delta}_r$-closed if and only if A is δ-I-closed.

Proof. Necessity - Suppose A is $\hat{\delta}_r$-closed. By Theorem 4.17, σcl(A) ⊂ $A_r^\wedge$ = A, since A is $\wedge_r$-set. Therefore A is δ-I-closed.

Sufficiency - The proof is follows from the fact that every δ-I-closed set is $\hat{\delta}_r$-closed.

Corollary 4.23 Let A be a $\wedge_r$-set of an ideal space (X, τ, I). Then A is δg-closed if and only if A is δ-I-closed.

Theorem 4.24 Let (X, τ, I) be an ideal space. If A is a $\hat{\delta}_r$-closed subset of X and A ⊂ B ⊂ σcl(A), then B is also $\hat{\delta}_r$-closed.

Proof. Since A ⊂B, σcl(A) ⊆ σcl(B). But σcl(B) ⊂ σcl(A). Hence σcl(A) = σcl(B). Hence B is $\hat{\delta}_r$-closed.

Theorem 4.25 Let (X, τ, I) be an ideal space and A be a $\hat{\delta}_r$-closed set. Then A ∪ (X–σcl(A)) is $\hat{\delta}_r$-closed.

Proof. Suppose that A is a $\hat{\delta}_r$-closed set. If U is any regularopen set such that A ∪ (X–σcl(A)) ⊂ U, then X–U ⊂ X– (A∪(X–σcl(A))) = σcl(A)–A. Since X–U is regularclosed and A is $\hat{\delta}_r$-closed, by Theorem 4.17, it follows that X–U = ϕ and so X=U. Hence X is the only regularopen set containing A ∪ (X–σcl(A)) and so A ∪ (X–σcl(A)) is $\hat{\delta}_r$-closed.

Remark 4.26 The following Example shows that the reversible implication is not true.

Example 4.27 In Example 3.4, let A = {a}. Then A ∪ (X–σcl(A)) = {a} ∪ (X–σcl({a})) = {a} ∪ (X–{a, c, d}) = {a} ∪ {b} = {a, b} is $\hat{\delta}_r$-closed but A = {a} is not $\hat{\delta}_r$-closed.

Theorem 4.28 For an ideal space (X, τ, I), the following are equivalent.

(i) Every $\hat{\delta}_r$-closed set is δ-I-closed.
(ii) Every singleton of X is regularclosed or δ-I-open.

Proof. (i) ⇒ (ii). Let x∈X. If {x} is not regularclosed, then A = X–{x} is not regularopen. Hence X is the only regularopen set containing A. Therefore A is $\hat{\delta}_r$-closed. Therefore by (i) A is δ-I-closed. Hence {x} is δ-I-open.
(ii) ⇒ (i). Let A be a $\hat{\delta}_r$-closed set and let x∈σcl(A), then we have the following cases.

Case (i). {x} is regularclosed. By Theorem 4.17, σcl(A)–A does not contain a non-empty regularclosed subset, This shows that {x} ∈ A.

Case (ii). {x} is δ-I-open, then {x} ∩ A ≠ ϕ. Hence {x} ∈ A. Thus in both cases {x} ∈ A and so A = σcl(A). That is A is δ-I-closed.

Theorem 4.29 In an ideal space (X, τ, I), for each x∈X, either {x} is regularclosed or $\{x\}^c$ is $\hat{\delta}_r$-closed set.

Proof. Suppose that {x} is not a regularclosed set in (X, τ, I). Then $\{x\}^c$ is not a regularopen set and the only regularopen set containing $\{x\}^c$ is X. Therefore $\{x\}^c$ is $\hat{\delta}_r$-closed set in (X, τ, I).

Corollary 4.30 In an ideal space (X, τ, I), for each x∈X, either {x} is semi-closed set or $\{x\}^c$ is $\hat{\delta}_r$-closed set.

Corollary 4.31 In an ideal space (X, τ, I), for each x∈X, either {x} is closed set or $\{x\}^c$ is $\hat{\delta}_r$-closed set.

Corollary 4.32 In an ideal space (X, τ, I), for each x∈X, either {x} is pre-closed set or $\{x\}^c$ is $\hat{\delta}_r$-closed set.

Corollary 4.33 In an ideal space (X, τ, I), for each x∈X, either {x} is α-closed set or $\{x\}^c$ is $\hat{\delta}_r$-closed set.

Theorem 4.34 If A and B are $\hat{\delta}_r$-closed sets in a topological space (X, τ, I), then A ∪ B is $\hat{\delta}_r$-closed set in (X, τ, I).

Proof. Suppose that A ∪ B ⊆ U, where U is any regularopen set in (X, τ, I) then A ⊆ U and B ⊆ U. By hypothesis σcl(A) ⊆ U and σcl(B) ⊆ U. Always σcl(A∪B) = σcl(A) ∪ σcl(B). Therefore, σcl(A∪B) ⊆ U. Thus A∪B is a $\hat{\delta}_r$-closed set in (X, τ, I).

Remark 4.35 If A and B are $\hat{\delta}_r$-closed sets in a topological space (X, τ, I), then A ∩ B is need not be $\hat{\delta}_r$-closed as shown in the following Example.

Example 4.36 Let X = {a, b, c, d}, τ = {X, ϕ, {b}, {c, d}, {b, c, d}} and I = {ϕ, {c}}. Let A = {a, c, d} and B = {b, c, d}. Then A and B are $\hat{\delta}_r$-closed sets. But A ∩ B = {a, c, d} ∩ {b, c, d} = {c, d} is not $\hat{\delta}_r$-closed.

Definition 4.37 A proper non-empty $\hat{\delta}_r$-closed subset A of an ideal space (X, τ, I) is said to be maximal $\hat{\delta}_r$-closed if any $\hat{\delta}_r$-closed set containing A is either X or A.

Example 4.38 In Example 3.4, let A = {a, c, d}. Then A is a maximal $\hat{\delta}_r$-closed subset of X.

Remark 4.39 Every maximal $\hat{\delta}_r$-closed set is $\hat{\delta}_r$-closed but, the converse is not hold as shown in the following Example.

Example 4.40 In Example 3.4, let A = {a, c}. Then A is $\hat{\delta}_r$-closed but it is not maximal.

Theorem 4.41 In an ideal space (X, τ, I), the following are true.

(i) Let F be a maximal $\hat{\delta}_r$-closed set and G be a $\hat{\delta}_r$-closed set then F ∪ G = X or G ⊂ F.

(ii) Let F and G be maximal $\hat{\delta}_r$-closed sets. Then F ∪ G = X or F = G.

Proof. (i) Let F be a maximal $\hat{\delta}_r$-closed set and G be a $\hat{\delta}_r$-closed set. If F ∪ G = X, then there is nothing to prove. Assume that F ∪ G ≠ X. Since union of two $\hat{\delta}_r$-closed set is $\hat{\delta}_r$-closed and since F is a maximal $\hat{\delta}_r$-closed set we have F ∪ G = X or F ∪ G = F. Hence F ∪ G = F and so G ⊂ F.
(ii) Let F and G be maximal $\hat{\delta}_r$-closed sets. If F ∪ G = X, then there is nothing to prove. Assume that F ∪ G ≠ X. Then by (i) F ⊂ G and G ⊂ F which implies F = G.

Theorem 4.42 If A is regularopen and $\hat{\delta}_r$ - closed then A is δ-I-closed

Proof. Suppose A is regularopen, $\hat{\delta}_r$ - closed set and A⊂A. Therefore, σcl(A)⊂A Hence A is δ-I-closed.

Remark 4.43 The following Example shows that the converse of Theorem 4.42 is not true.

Example 4.44 In Example 3.7, let A= {c, d}. Then A is δ-I-closed and hence $\hat{\delta}_r$ – closed but not regularopen.

Theorem 4.45 If A is regularopen and $\hat{\delta}_r$–closed, then A is regularclosed and hence clopen.

Proof. Suppose A is regularopen, $\hat{\delta}_r$–closed and A⊂A, we have σcl(A)⊂A and hence cl(A)⊂A. Therefore A is closed. Since A is regularopen, A is open. Now, cl(int(A)) = cl(A)=A.

Corollary 4.46 If A is regularopen and $\hat{\delta}_r$ – closed, then A is regularclosed and hence α – open and α – closed.

Corollary 4.47 If A is regularopen and $\hat{\delta}_r$ – closed, then A is regularclosed and hence semi-open, semi-closed.

Corollary 4.48 Let A be a regularopen and $\hat{\delta}_r$ – closed in X. Suppose that F is δ-I- closed in X. Then A∩F is a $\hat{\delta}_r$ – closed in X.
Proof. Let A be a regularopen and $\hat{\delta}_r$ – closed set and F be δ-I-closed in (X, τ, I). By Theorem 4.42, A is δ-I-closed. So A∩F is δ-I-closed and hence A∩F is $\hat{\delta}_r$ –closed set in X.

Corollary 4.49 Let A be a regularopen and $\hat{\delta}_r$ – closed set in X. Suppose F is δ-closed in X. Then A∩F is $\hat{\delta}_r$ - closed in X.

Corollary 4.50 Let A be a regularopen and $\hat{\delta}_r$ – closed. Suppose F is $\hat{\delta}_s$- closed in X. Then A∩F is $\hat{\delta}_r$ – closed in X.

Proof. The proof is follows from the fact that the intersection of a δ-I-closed set and a $\hat{\delta}_s$ - closed set is $\hat{\delta}_s$-closed and every $\hat{\delta}_s$-closed set is $\hat{\delta}_r$ – closed (By Theorem 4.11[17] and By Theorem 3.2).

Theorem 4.51 In an ideal topological space (X, τ, I) if RO(τ) = {X, ϕ}, then every subset of X is a $\hat{\delta}_r$ – closed set.

Proof. Let X be a topological space and RO(τ) = {X, ϕ}. Let A be any subset of X. Suppose A = ϕ, then A is trivially $\hat{\delta}_r$-closed set. Suppose A ≠ ϕ. Then X is the only regularopen set containing A and so σcl(A)⊂X. Hence A is $\hat{\delta}_r$ – closed set in X.

Remark 4.52 The converse of Theorem 4.51 need not be true in general as shown in the following example.

Example 4.53 Let X = {a, b, c, d}, τ = {X, ϕ, {a}, {b}, {a, b}, {b, c, d}} and I = {ϕ, {a}}. Then $\hat{\delta}_r$-closed sets equals P(X). But RO (τ) = {X, ϕ, {a}, {b, c, d}}.

Theorem 4.54 In an ideal space (X, τ, I), RO(X, τ)⊂{F⊂X:F^c⊂$\tau_{\delta\text{-}I}$} if and only if every subset of (X, τ, I) is $\hat{\delta}_r$ – closed.

Proof. Necessity – Suppose RO(X, τ)⊂{F⊂X:F^c∈$\tau_{\delta\text{-}\tau}$}. Let A be any subset of (X, τ, I) such that A⊂U, where U is any regularopen set in X. Then U∈RO(X, τ) ⊂{F⊂X: F^c∈$\tau_{\delta\text{-}I}$}. That is U∈{F⊂X: F^c∈$\tau_{\delta\text{-}I}$}. Thus U is δ-I-closed. Then σcl(U) = U. Also σcl(A)⊂σcl(U) = U. Hence A is $\hat{\delta}_r$ – closed set in X.

Sufficiency - Suppose that every subset of (X, τ, I) is $\hat{\delta}_r$ – closed. Let U∈RO(X, τ). Since U⊂U and U is $\hat{\delta}_r$ – closed, we have σcl(U)⊂U. Thus σcl(U) = U and U∈{F⊂X: F^c∈$\tau_{\delta\text{-}I}$}.
Therefore RO(X, τ)⊂ {F⊂X: F^c∈$\tau_{\delta\text{-}I}$}.

Theorem 4.55 In an ideal space (X, τ, I) if A is a $\hat{\delta}_r$ – open set then G = X, whenever G is regularopen and σint(A)∪(X–A)⊂G.

Proof. Let A be a $\hat{\delta}_r$ – open set. Suppose G is regularopen set such that σint(A)∪(X–A)⊂G. Then X–G⊂X–(σint(A)∪(X–A)) = (X–σint(A))∩A = (X–σint(A)) – (X–A) = σcl(X–A) – (X–A). Since X–A is $\hat{\delta}_r$ – closed, by Theorem 4.2, X–G = ϕ. Therefore G = X.

Remark 4.56 The following Example shows that the reversible implication is not hold.

Example 4.57 In Example 4.36, let A = {a, c, d}. Then σint(A)∪(X–A) = {c, d}∪{b} = {b, c, d} and X is the only regularopen set containing σint(A)∪(X–A). Since A is regularclosed and A⊄σint(A) = {c, d}, A is not $\hat{\delta}_r$ – open.

Theorem 4.58 Let (X, τ, I) be an ideal space. A subset A⊂X is $\hat{\delta}_r$ – open if and only if F⊂σint(A) whenever F is regularclosed and F⊂A.

Proof. Necessity - A is $\hat{\hat{\delta}}_r$ – open. Let F be a regularclosed set contained in A. Then $\sigma cl(X–A) \subset X–F$ and $F \subset X–\sigma cl(X–A)$.

Sufficiency - Suppose $X–A \subset U$ and U is regularopen. By hypothesis, $X–U \subset \sigma int(A)$. Which implies $\sigma cl(X–A) \subset U$. Therefore X–A is $\hat{\hat{\delta}}_r$ – closed and hence A is $\hat{\hat{\delta}}_r$ – open.

Theorem 4.59 If A is a $\hat{\hat{\delta}}_r$ – closed set in an ideal space (X, τ, I), then $\sigma cl(A)–A$ is $\hat{\hat{\delta}}_r$ – open.

Proof. Since A is $\hat{\hat{\delta}}_r$ – closed, by Theorem 4.2, the only regular closed set contained in $\sigma cl(A)–A$ is ϕ. Therefore by Theorem 4.58, $\sigma cl(A)–A$ is $\hat{\hat{\delta}}_r$ – open.

Remark 4.60 The following Example shows that the reverse direction of Theorem 4.59 is not true.

Example 4.61 In Example 4.36, let A = {c, d}. Then $\sigma cl(A)–A$ = {a, c, d} – {c, d} = {a} is $\hat{\hat{\delta}}_r$ – open but A = {c, d} is not $\hat{\hat{\delta}}_r$ – closed.

Theorem 4.62 Let (X, τ, I) be an ideal space and $A \subset X$. If A is $\hat{\hat{\delta}}_r$-open and $\sigma int(A) \subset B \subset A$, then B is $\hat{\hat{\delta}}_r$ – open.

Proof. Since $\sigma int(A) \subset B \subset A$, we have $\sigma int(A) = \sigma int(B)$. Suppose F is regular closed set contained in B, then $F \subset A$. Since A is $\hat{\hat{\delta}}_r$ – open, by Theorem 4.58, $F \subset \sigma int(A) = \sigma int(B)$. Therefore again by Theorem 4.58, B is $\hat{\hat{\delta}}_r$ – open.

Definition 4.63 In an ideal space (X, τ, I), a subset A of X is said to be $\delta_I\wedge_r$ – set if $A^{\wedge}_r \subset F$ whenever $A \subset F$ and F is δ-I-closed. A subset A of X is called $\delta_I\vee_r$ – set if X–A is $\delta_I\wedge_r$ – set.

Lemma 4.64 Let (X, τ) be a space then $(X–A)^{\wedge}_r = X–A^{\vee}_r$ for every subset A of X.

Theorem 4.65 A subset A of an ideal space (X, τ, I) is an $\delta_I\vee_r$ – set if and only if $U \subset A^{\vee}_r$ whenever $U \subset A$ and U is δ-I-open.

Proof. Necessity - Let A be an $\delta_I\vee_r$ – set. Suppose U is a δ-I-open set such that $U \subset A$. Since X–A is a $\delta_I\wedge_r$ – set, $(X-A)^{\wedge}_r \subset X–U$ and by Lemma 4.64, $X–A^{\vee}_r \subset X–U$. Therefore $U \subset A^{\vee}_r$.

Sufficiency - Suppose that $X–A \subset F$ and F is δ-I-closed, by hypothesis $X–F \subset A^{\vee}_r$ and therefore $X–A^{\vee}_r \subset F$. Therefore by Lemma 4.64, $(X–A)^{\wedge}_r \subset F$. Therefore, X–A is a $\delta_I\wedge_r$ – set and hence A is $\delta_I\vee_r$ – set.

Theorem 4.66 Let (X, τ, I) be an ideal space. Then for each $x \in X$, {x} is either δ-I-open or a $\delta_I\vee_r$ – set.

Proof. Suppose {x} is not δ-I-open for some $x \in X$. Then X–{x} is not δ-I-closed. Therefore the only δ-I-closed set containing X–{x} is X. Therefore X–{x} is a $\delta_I\wedge_r$ – set and hence {x} is $\delta_I\vee_r$ – set.

Theorem 4.67 Let A be a $\delta_I\vee_r$ – set in (X, τ, I). Then for every δ-I-closed set F such that $A^{\vee}_r \cup (X–A) \subset F$, F equals X.

Proof. Let A be a $\delta_I\vee_r$ – set. Suppose F is a δ-I-closed set such that $A^{\vee}_r \cup (X–A) \subset F$. Then $X–F \subset X–(A^{\vee}_r \cup (X–A)) = (X–A^{\vee}_r) \cap A$. Since A is a $\delta_I\vee_r$ – set and X–F is a δ-I-open subset of A, by Theorem 4.65, $X–F \subset A^{\vee}_r$. Also, $X–F \subset X–A^{\vee}_r$. Therefore $X–F \subset A^{\vee}_r \cap (X–A^{\vee}_r) = \phi$. Hence F=X.

Theorem 4.68 Let A be a $\delta_I\vee_r$ – set in an ideal space (X, τ, I). Then $A^{\vee}_r \cup (X–A)$ is δ-I-closed if and only if A is a $\vee_r$ – set.

Proof. Necessity - Let A be a $\delta_I\vee_r$ – set in (X, τ, I). If $A^{\vee}_r \cup (X–A)$ is δ-I-closed then by Theorem 4.67, $A^{\vee}_r \cup (X–A) = X$ and so $A \subset A^{\vee}_r$. Therefore, $A = A^{\vee}_r$ which implies that A is a $\vee_r$ – set.

Sufficiency - suppose A is a $\vee_r$ – set. Then $A = A^{\vee}_r$ and so $A^{\vee}_r \cup (X–A) = A \cup (X–A) = X$ is δ-I-closed.

Theorem 4.69 Let A be a subset of an ideal space (X, τ, I) such that $A^{\vee}_r$ is δ-I-closed. If X is the only δ-I-closed set containing $A^{\vee}_r \cup (X–A)$, then A is a $\delta_I\vee_r$ – set.

Proof. Let U be a δ-I-open set contained in A. Since $A^{\vee}_r$ is δ-I-closed, $A^{\vee}_r \cup (X–U)$ is δ-I-closed. Also, $A^{\vee}_r \cup (X–A) \subset A^{\vee}_r \cup (X–U)$. By hypothesis, $A^{\vee}_r \cup (X–U) = X$. Therefore $U \subset A^{\vee}_r$. Therefore A is a $\delta_I\vee_r$ – set.

Theorem 4.70 In a $T_{1/2}$ (resp. T_I) - space every $\hat{\hat{\delta}}$ - closed set is closed (resp. *-closed).

Proof. Let X be $T_{1/2}$ (resp. T_I) – space. Let A be $\hat{\hat{\delta}}$-closed set of X. Since every $\hat{\hat{\delta}}$-closed set is g(resp. I_g) – closed and X is $T_{1/2}$ (resp. T_I) – space, A is closed (resp. *-closed).

Theorem 4.71 In an ideal space (X, τ, I), the following are equivalent

a) Every δg – closed set is * - closed
b) (X, τ, I) is a T_I – space
c) Every $\hat{\hat{\delta}}$ – closed set is * - closed.

Proof. (1)⇒(2). Let $x \in X$, If {x} is not closed, then X is the only open set containing X–{x} and so X–{x} is δg – closed. By hypothesis X–{x} is *-closed. Therefore {x} is *-open. Thus every singleton set in X is either closed or *-open. By Theorem 3.3 [6], (X, τ, I) is a T_I – space.

(2)⇒(1) Let A be a δg – closed set. Since every δg – closed set is g – closed [8] and hence I_g – closed, A is I_g – closed set. By hypothesis A is * - closed.

(2) ⇒(3) Let A be a $\hat{\hat{\delta}}$-closed set. Since every $\hat{\hat{\delta}}$-closed set is g – closed and hence I_g – closed set, A is I_g – closed. By hypothesis A is * - closed.

(3) ⇒(2) Let $x \in X$. If {x} is not closed, then X is the only open set containing X–{x} and so X–{x} is $\hat{\hat{\delta}}$ – closed. By hypothesis, X–{x} is * - closed. Thus {x} is * -open. Therefore every singleton set in X is either closed or * - open. By Theorem 3.3 [6], (X,τ, I) is a T_I–space.

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