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Pythagorean triplets (alternative approach, algebraic operations, dual of given triplets, and new observations)

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Abstract
In this paper we have introduced a new approach tantamounting the regular one. It helps derive some additional algebraic properties. The novel concept of finding dual triplets of a given triplet and its generalization is an important feature of this note. Some new observations on ratios of corresponding sides of consecutive odd Pythagorean triplets and numerical properties related to 'Jha sequence' and 'NSW sequence' are explicitly demonstrated and derived in three annexure.

Keywords
(1) Primitive Pythagorean triplets
(2) Dual triplets of a given triplet
(3) Ratio of corresponding sides
(4) Jha sequence
(5) Fermat Family

Introduction

This paper contains vivid properties and observations derived as an extension to the basic properties of Pythagorean triplets.

The first part introduces the pattern and nature of Pythagorean triplet in a new form taking it to angels and amplitudes corresponding to a given triplet. This helps understand uniqueness of angles associated to given triplets. Also this presentation when applied to pursue algebraic operations like addition and subtraction gives interesting results of finding two different Pythagorean triplets with the same hypogenous which is, in fact, the product of two hypotenuses of two distinct primitive triplets.

In the second part we have taken the rations of corresponding sides of consecutively ordered odd Pythagorean triplets. These ratios exhibit a special pattern of recurrence relation. The result of their rate of convergence is also highly note-worthy.

In the third section, we have introduced a new concept by relaxing the restrictions on a, b and $h$and allowing them to be in the set Q^+ U $\{0\}$. As a result to this we introduce dual triplets of a given triplet. All the dual triplets exhibit parallel properties to that of Pythagorean triplets is a fascinating exploration to generalization. Also, we have explicitly discussed about the total number of dual triplets associated with a given triplet.

In the last - the fourth section, we have introduced two new observations on the nature of sides of some triplets and have also proved them in three different annexures. This paper, in its last part, introduces 'Jha-sequence', a natural sequence which arises from the geometry of Fermat family of right triangles satisfying Pythagorean property.

Introduction – Nature of Primitive Triplet

Prior to going into detail of each section, we introduce some basic set-up of Pythagorean triplet often used in this paper.

We call a triplet (a, b, h) a primitive Pythagorean triplet, if the following conditions on the sides a, b, and h are satisfied
(1) $a, b \text{ and } h \in N$
(2) $g.c.d \text{ of } a \text{ and } b = (a, b) = 1$
(3) $a < b < h,$
(4) $a^2 + b^2 = h^2$ ------ (1)
We define a set T of primitive triplets as defined above,
T= $\{(a, b, h)| \text{ satisfying } (1) \text{above}\}$

Comments:

(i) If a triplet $(b_1, a_1, h_1)$with $b_1 > a_1$satisfies conditions $(1), (2), and\ (4)$ as shown above, then we can put it as(a_1, b_1, h_1); as it satisfies conditions $(1)\ to\ (4)$ shown above.

$e.g.$Instead of writing $(4, 3, 5)$ a triplet not satisfying conditions(3), we put it in the pattern $(3, 4, 5)$ which conforms with all the conditions shown above. $(3, 4, 5)$ is the first and the only primitive triplet which falls in the set of triplets of Pythagorean family, Plato family, and Fermat family.

(ii) $(a, b) = 1$ Implies that both $a \text{ and } b$ cannot be even and hence h is always odd.

(iii) It is clear from above comment that $d_1 = h - a_1 \geq 2 \text{ and } d_1 \in N \text{ and} d_2 = h + a_1 \geq 8$ and $d_2 \in N$

(iv) Let $t_1 = (a_1, b_1, h_1)$ and $t_2 = (a_2, b_2, h_2)$ be two distinct member triplets of T. We will establish the fact that there exists two distinct primitive triplets $t_3 = (a_3, b_3, h^*)$ and $t_4 = (a_4, b_4, h^*) \in$ T;

such that $h^* = \sqrt{{a_3}^2 + {b_3}^2} = \sqrt{{a_4}^2 + {b_4}^2}$

$e.g$: $t_1 = (3,4,5)$, $t_2 = (5,12,13)$ and $(h_1)(h_2) = 5 \times 13 = 65$.
We have
$t_3 = (16, 63, 65)$ and $t_4 = (33, 56, 65)$ are two primitive triplets with the same hypotenuse.

(v) For some $t \in$T, if a is an even integer then b cannot be a prime.
We prove it as follows.
Let$t_1 \in$ T; T $= \{(a, b, h)|(a, b, h) \text{is a primitive triplet.}\}$
If $a \in N$ is an even integer then $b \in N$ cannot be a prime.
By the Pythagorean properties,
$a^2 + b^2 = h^2$ and$h$ is always an odd integer.
$\therefore b^2 = h^2 - a^2$
$\therefore b^2 = (h - a)(h + a)$
$\therefore b = \sqrt{(h - a)(h + a)} \geq 4$ (Reference of comment above (iii)) ----- (2)
$a \neq 1 \text{ and } a \neq 2\ \therefore a \geq 3$ also $h - a \geq 2$
$h \geq 5$ and $a \geq 3 \therefore h + a \geq 8$
Since h is an odd integer and a is an even, so $h - a$ and$h + a$ are both odd and$h + a \geq 9$ and$h - a \geq 3$.
$\therefore b^2 = (h - a)(h + a) \geq 27$ is an odd integer.
$\therefore b = \sqrt{h - a} \cdot \sqrt{h + a}$
$\therefore b$ has two factors and none of which is unity.
$\therefore b$ cannot be a prime.

The alternative proof regarding the same is given in annexure-1.

Geometric Approach

Let (a, b, h) be a Primitive triplet which conforms to the format described above.The basic structure of the format helps us represent the primitive triplets in the first quadrant of R^2 perpendicular set of axis.
For$(a, b, h), a, b \text{ and } h \in N$.
The coordinate $(a, b) = \bar{v}$, a vector inR^2
and $\|\bar{v}\| = \sqrt{a^2 + b^2} = h$.
We have developed another system of representing a given primitive triplet. This pattern helps derive some algebraic structures.
For any primitive triplet (a, b, h), we have $a \in N - \{1, 2\}$ and $b > 3$ are such that $a < b$ and for a fixed ratio say,

$\lambda = \frac{a}{b}$, we have $0 < \lambda = \frac{a}{b} < 1$.
Now, we define $cot^{-1}: (0,1) \rightarrow \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \ni$ if $\lambda = \frac{a}{b} \in (0,1)$
then, $\cot^{-1} \lambda = \cot^{-1}\left(\frac{a}{b}\right) = \theta$ say, and$\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
In continuation to this, we define $f: \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \rightarrow N$ such that if $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$;
then $f(\theta) = h = \sqrt{a^2 + b^2}$. where $\cot^{-1}(\frac{a}{b}) = \theta$, ------- (3)
Combining these facts we write that a primitive triplet $(a, b, h) \approx (\theta, f(\theta))$, where $\cot\theta = \frac{a}{b}$. $\frac{\pi}{4} < \theta < \frac{\pi}{2}$as $0 < \frac{a}{b} < 1$ and$f(\theta) = h = \sqrt{a^2 + b^2} \in N$
With this definition, we can derive that for each triplet from T, the corresponding $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is unique.
For every triplet (a, b, h), $\theta = \cot^{-1}(\frac{a}{b})$ is unique.
i.e., for $t_1 = (a_1, b_1, h_1)$ and $t_2 = (a_2, b_2, h_2)$ with $a_1 \neq a_2$ then $\theta_1 \neq \theta_2$.
and similarly, $b_1 \neq b_2$then $\theta_1 \neq \theta_2$.
Also $\left({a_1}/{b_1}\right) \neq \left({a_2}/{b_2}\right)$, as a consequence of definition of primitive triplet
In order to prove uniqueness, we define F: T $\rightarrow \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \ni$ $if\ t_1 \in$ T then $F(t_1) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
Let $F(t_1) = \theta_1$. This $\theta_1 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is such that $\cot\theta_1 = \frac{a_1}{b_1}$ and
$f(\theta_1) = \sqrt{{a_1}^2 + {b_1}^2} = h_1 \in N$ as defined by (3).
For$t_1 = (a_1, b_1, h_1), t_2 = (a_2, b_2, h_2) \in$ T
If$t_1 \neq t_2$ then, $F(t_1) \neq F(t_2)$. Let $F(t_1) = \theta_1$ and $F(t_2) = \theta_2$ and both
$\theta_1, \theta_2 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cot\theta_1 = \frac{a_1}{b_1}$ and $\cot\theta_2 = \frac{a_2}{b_2}$.,
If $(a_1, b_1) \neq (a_2, b_2)$ and $(a_2, b_2) \neq k(a_1, b_1), k \in N$ then, $\theta_1 \neq \theta_2$.
Also this function F: T $\rightarrow \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is an into function. This can be clearly established by the definition of primitive triplets.

Algebraic Operations:

In this section we define two algebraic properties on primitive triplets. The prime purpose is to derive that for $t_1, t_2 \in$ T and $t_1 \neq t_2$, we have some $t_3 = (a_3, b_3, h_3)$,$t_4 = (a_4, b_4, h_4)$ ∈T with the following important properties,

1) $h_3 = h_4$

2) $h_3 = h_4$=$h_1 . h_2$, where $t_3 = (a_3, b_3, h_3)$ $t_4 = (a_4, b_4, h_4)$

As F: T $\rightarrow \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ is one-one and into functions; let

$F(t_3) = \theta_3$ and $F(t_4) = \theta_4$ with

$\theta_3, \theta_4 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. $\cot\theta_3 = \frac{a_3}{b_3}$ and $\cot\theta_4 = \frac{a_4}{b_4}$.

If $(a_1, b_1) \neq (a_2, b_2)$ then $\theta_3 \neq \theta_4$ (uniqueness). Say $\theta_4 > \theta_3 \Longmapsto \cot\theta_4 < \cot\theta_3$.

In continuation with the above, we introduce two algebraic operations.

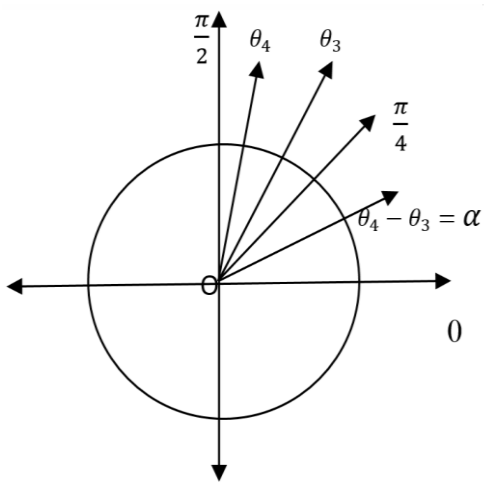
First we define their subtraction.

As $\theta_4 > \theta_3$, let $\theta_4 - \theta_3 = \alpha$ and $0 < \alpha < \frac{\pi}{4}$.

$$\therefore \cot\alpha = \cot(\theta_4 - \theta_3) = \frac{\cot\theta_4 \cdot \cot\theta_3 + 1}{\cot\theta_3 - \cot\theta_4}$$

$$= \frac{\left(\frac{a_4}{b_4}\right)\left(\frac{a_3}{b_3}\right) + 1}{\left(\frac{a_3}{b_3}\right) - \left(\frac{a_4}{b_4}\right)}$$

$$= \frac{a_3 a_4 + b_3 b_4}{a_3 b_4 - a_4 b_3} > 1$$



Therefore, by definition we cannot have a triplet for which $\cot\alpha > 1$. as explained in the comment (1). We reconstruct the same triplets in the form, $(a_3 b_4 - a_4 b_3, a_3 a_4 + b_3 b_4, h)$,

Where $h = \sqrt{(a_3 b_4 - a_4 b_3)^2 . (a_3 a_4 + b_3 b_4)^2}$

= $\sqrt{(a_3{}^2 + b_3)^2 (a_4{}^2 + b_4)^2}$

=$\sqrt{h_3{}^2 . h_4{}^2}$

$\therefore$ h = $h_3 \cdot h_4$

And $\cot\alpha = \frac{a_3 b_4 - a_4 b_3}{a_3 a_4 + b_3 b_4} < 1$, where $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$

This establishes the fact that corresponding to two primitive triplets, $t_3 = (a_3, b_3, h_3)$ and $t_4 = (a_4, b_4, h_4)$, there exists a primitive triplets with hypogenoush= $h_3 \cdot h_4$.

$i.e$, This triplet is $(a_3 b_4 - a_4 b_3,\ a_3 a_4 + b_3 b_4, \text{h} = h_3 \cdot h_4)$. ---- (4)

Angles- Additive Property:

In reference to the triplets $t_3 = (a_3, b_3, h_3)$ and $t_4 = (a_4, b_4, h_4)$described above

And corresponding anglesθ_4, and θ_3,we have $\theta_4 > \theta_3$, Let $\theta_4 + \theta_3 = \beta$ and

$\frac{\pi}{2} < \theta_4 + \theta_3 = \beta < \pi$.($\because \theta_3, \theta_4 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$)

We have three cases:

1. $\frac{\pi}{2} < \theta_4 + \theta_3 < \frac{3\pi}{4}$

2. $\theta_4 + \theta_3 = \frac{3\pi}{4}$

3. $\frac{3\pi}{4} < \theta_4 + \theta_3 < \pi$

Case -1. For$\frac{\pi}{2} < \theta_4 + \theta_3 = \beta < \frac{3\pi}{4}$. we have $-1 < \cot(\theta_3 + \theta_4) < 0$. For this case we find its geometric equivalent in$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

$$\therefore \frac{\pi}{2} - \pi < \theta_4 + \theta_3 - \pi < \frac{3\pi}{4} - \pi$$

$$\Longmapsto -\frac{\pi}{2} < \theta_4 + \theta_3 - \pi < -\frac{\pi}{4}$$

$$\Longmapsto \frac{\pi}{4} < \pi - (\theta_4 + \theta_3) < \frac{\pi}{2}$$

Now, $\cot(\theta_3 + \theta_4) = |\cot(\pi - (\theta_3 + \theta_4))|$

$$= \left|\frac{\cot\theta_4 \cdot \cot\theta_3 - 1}{\cot\theta_3 + \cot\theta_4}\right|$$

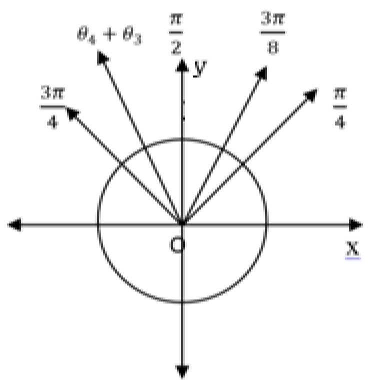
$$= \left|\frac{\left(\frac{a_4}{b_4}\right)\left(\frac{a_3}{b_3}\right) - 1}{\left(\frac{a_3}{b_3}\right) + \left(\frac{a_4}{b_4}\right)}\right| = \left|\frac{a_3 a_4 - b_3 b_4}{a_3 b_4 + a_4 b_3}\right| < 1.$$

Therefore, our triplets will be $(|a_3 a_4 - b_3 b_4|, (a_3 b_4 + a_4 b_3),\ h)$, ----- (5)

Where $h = \sqrt{(a_3 b_4 - a_4 b_3)^2 . (a_3 a_4 + b_3 b_4)^2}$ =$\sqrt{(a_3{}^2 + b_3)^2 (a_4{}^2 + b_4)^2}$

=$\sqrt{h_3{}^2 . h_4{}^2}$

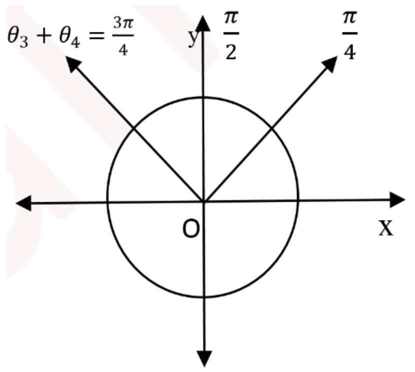
$\therefore$ h = $h_3 \cdot h_4$



Case-2 For $\theta_3 + \theta_4 = \frac{3\pi}{4}$Let θ_3 correspond to the primitive triplet (a_3, b_3, h_3) and θ_4 correspond to the primitive triplet (a_4, b_4, h_4) then $sin\theta_3 = \frac{b_3}{h_3}$, $cos\theta_3 = \frac{a_3}{h_3}$ and $sin\theta_4 = \frac{b_4}{h_4}$, $cos\theta_4 = \frac{a_4}{h_4}$ with $\frac{b_3}{h_3}, \frac{a_3}{h_3}, \frac{b_4}{h_4}$

and $\frac{a_4}{h_4} \in Q^+$ and $0 < \frac{b_3}{h_3}, \frac{a_3}{h_3}, \frac{b_4}{h_4}, \frac{a_4}{h_4} < 1$

Now $\sin(\theta_3 + \theta_4) = \sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ is an irrational number which cannot be
equivalent to $sin\theta_3 \cdot cos\theta_4 + cos\theta_3 \cdot sin\theta_4$,
which is an expression in Q^+.
It is a contradiction. Hence, there cannot be two primitive triplets corresponding to the angles $\theta_3 and\ \theta_4$for which $\theta_3 + \theta_4 = \frac{3\pi}{4}$.



Case–3 For $\frac{3\pi}{4} < \theta_3 + \theta_4 < \pi$ and if at least one θ_3 *or* θ_4 lies between ($\frac{3\pi}{8}, \frac{\pi}{2}$) then

$$\frac{3\pi}{4} < \theta_3 + \theta_4 < \pi$$

$$\Longrightarrow -\frac{3\pi}{4} > -(\theta_3 + \theta_4) > -\pi$$

$$\therefore \frac{\pi}{4} > \pi - (\theta_3 + \theta_4) > 0$$

$$\therefore 0 < \pi - (\theta_3 + \theta_4) < \frac{\pi}{4}$$

x

Now$\cot(\theta_3 + \theta_4) = |\cot(\pi - (\theta_3 + \theta_4))| < 1$

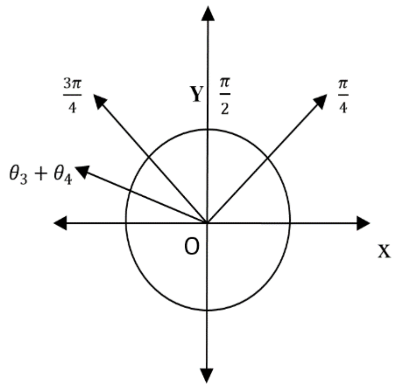
$\therefore \cot(\theta_3 + \theta_4) = \left|\frac{\cot\theta_4 \cdot \cot\theta_3 - 1}{\cot\theta_3 + \cot\theta_4}\right| =$

$\left|\frac{\left(\frac{a_4}{b_4}\right)\left(\frac{a_3}{b_3}\right) - 1}{\left(\frac{a_3}{b_3}\right) + \left(\frac{a_4}{b_4}\right)}\right| = \left|\frac{a_3 a_4 - b_3 b_4}{a_3 b_4 + a_4 b_3}\right| < 1.$

Therefore our triplets will be
$(|a_3 a_4 - b_3 b_4|,\ (a_3 b_4 + a_4 b_3), h)$, where

$$h = \sqrt{(a_3 b_4 - a_4 b_3)^2 . (a_3 a_4 + b_3 b_4)^2}$$
$$= \sqrt{(a_3{}^2 + b_3)^2 (a_4{}^2 + b_4)^2}$$
$$= \sqrt{h_3{}^2 + h_4{}^2}$$
$$\therefore \ \text{h} = h_3 \cdot h_4$$

All the three cases discussed above establish existence of two distinct triplets with the same hypogenous which is a product of hypotenuses of two distinct primitive triplets.



6 About the Ratios of Corresponding Sides:

Let us consider some triplets of T. let N = N_1 U N_2, where
$N_1 = \{2k + 1 | k \in N\}$ and
$N_2 = \{2\ k\ |\ k \in N\}$, let $T_1 = \{(a_i, b_i, h_i) | a_i \in N_1\}$; we call them an infinite set of odd triplets and
$T_2 = \{(a_i, b_i, h_i) | a_i \in N_2\}$; we call them an infinite set of even triplets.
And $T_1, T_2 \subset T$
Now we have $a_i \in N_1$ then $b_i \in N_2$ and$h_i \in N_1$ for all $i \in N$
Let us consider primitive triplets from the set T_1, for all $n \in$ N. We have (a_n, b_n, h_n).
[Say, we have some consecutive odd triplets of the set T_1 as follow. Let us consider the ratios of their corresponding sides.]

	a_n	b_n	h_n	$\frac{a_{n+1}}{a_n}$	$\frac{b_{n+1}}{b_n}$	$\frac{h_{n+1}}{h_n}$ for $n \in$ N
(1)	3	4	5	5/3	12/4	13/5
(2)	5	12	13	7/5	24/12	25/13
(3)	7	24	25	9/7	40/24 (=5/3)	41/25
(4)	9	40	41	11/9	60/40	61/41
(5)	11	60	61	13/11	84/60 (=7/5)	85/61
(6)	13	84	85	15/13	112/84	113/85
(7)	15	112	113	17/15	114/112 (=9/7)	145/113
(8)	17	114	145	19/17	180/114	181/145
(9)	19	180	181	21/19	220/180 (=11/9)	221/181
(10)	21	220	221	23/21	264/220	265/221

Now for some $n \in$ N, we have ratios as found above$\frac{a_{n+1}}{a_n}, \frac{b_{n+1}}{b_n}, \frac{h_{n+1}}{h_n}$

For these ratios we have following observations:

1.$\frac{a_{n+1}}{a_n} < \frac{h_{n+1}}{h_n} < \frac{b_{n+1}}{b_n}$

2. $\frac{a_{n+1}}{a_n} = \frac{b_{2n+2}}{b_{2n+1}}$

3. Let $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \alpha, \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \gamma, \lim_{n \to \infty} \frac{h_{n+1}}{h_n} = \beta$ then all $\alpha, \beta\ and\ \gamma$ tends to 1 and also $\alpha < \beta < \gamma$. In fact $|1 - \gamma| < |1 - \beta| < |1 - \alpha|$.

In the all above cases, we have considered a fact that $b_n = \frac{(a_n)^2 - 1}{2}, h_n = \frac{(a_n)^2 + 1}{2} \Longrightarrow b_n < h_n$.

Givena_n anda_{n+1} both are odd consecutive, so for $n \in$ N,
We have $a_n = 2n + 1 \Longrightarrow a_{n+1} = 2n + 3$.

Now, $\frac{a_{n+1}}{a_n} = \frac{2n+3}{2n+1} = 1 + \frac{2}{2n+1}$, so $\alpha = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

And $\frac{b_{n+1}}{b_n} = \frac{\{(2(n+1)+1)^2-1\}/2}{\{(2n+1)^2-1\}/2} = \frac{(2n+3)^2-1}{(2n+1)^2-1} = \frac{4n^2+12n+8}{4n^2+4n}$

$= \frac{(n+2)}{n} = 1 + \frac{2}{n}$.

so, $\gamma = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$.

Also for $\frac{h_{n+1}}{h_n} = \frac{\{(2(n+1)+1)^2+1\}/2}{\{(2n+1)^2+1\}/2} = \frac{(2n+3)^2+1}{(2n+1)^2+1} = \frac{4n^2+12n+10}{4n^2+4n+2} = 1 + \frac{(4n+4)}{2n^2+2n+1}$.

So, $\beta = \lim_{n\to\infty} \frac{h_{n+1}}{h_n} = 1$.

Now consider, $\frac{b_{n+1}}{b_n} - \frac{h_{n+1}}{h_n} = \left(1+\frac{2}{n}\right) - (1 + \frac{(4n+4)}{2n^2+2n+1}$

$$= \frac{2}{n} - \frac{(4n+4)}{2n^2+2n+1}$$

$= \frac{(4n^2+4n+2-4n^2-4n)}{n(2n^2+2n+1)} = \frac{2}{n(2n^2+2n+1)} > 0, for\ all\ n \in$ N

Therefore, $\frac{b_{n+1}}{b_n} > \frac{h_{n+1}}{h_n}$ ------- (7)

Also, for $\frac{h_{n+1}}{h_n} - \frac{a_{n+1}}{a_n} = (1 + \frac{(4n+4)}{2n^2+2n+1}) - (1 + \frac{2}{2n+1})$

$= \frac{(4n+4)}{2n^2+2n+1} - \frac{2}{2n+1} = \frac{4n^2+8n+2}{(2n^2+2n+1)(2n+1)} > 0$.

Therefore, $\frac{h_{n+1}}{h_n} > \frac{a_{n+1}}{a_n}$ ------- (8)

From (7) and (8), we have $\frac{a_{n+1}}{a_n} < \frac{h_{n+1}}{h_n} < \frac{b_{n+1}}{b_n} \Longrightarrow \alpha < \beta < \gamma$.

Next to show that $\frac{a_{n+1}}{a_n} = \frac{b_{2n+2}}{b_{2n+1}}$

Let R.H.S.= $\frac{b_{2n+2}}{b_{2n+1}} = \frac{\{(2(2n+2)+1)^2-1\}/2}{\{(2(2n+1)+1)^2-1\}/2} = \frac{(4n+5)^2-1}{(4n+3)^2-1}$

$= \frac{(2n+3)(n+1)}{(2n+1)(n+1)} = \frac{(2n+3)}{(2n+1)} = \frac{a_{n+1}}{a_n}$= L.H.S.

$\therefore \frac{a_{n+1}}{a_n} = \frac{b_{2n+2}}{b_{2n+1}}$ ------- (9)

6.1 Extended Version of Pythagorean Property and Duals of a Given Triplet.

It is known that Pythagorean property holds for some members of the set N. In order to explore more properties, we allow the sides $a, b\ and\ h$ to be the members of the set $Q^+ \cup \{0\}$. We, at this stage modify our norms on the primitive triplet like (a, b, h).

We continue with some defining characteristics of a primitive triplet and add a few to it.

For some $a, b\ and\ h$ all member of the set $Q^+ \cup \{0\}$,we introduce E-Pythagorean triplet satisfying the following characteristics.

1. $a, b\ and\ h \in Q^+ \cup \{0\}$
2. $a < b \le h$
3. $(a, b) = 1$
4. $a^2 + b^2 = h^2$

As a result to all above, we allow a primitive triplet like $(0, a, a)$for $a \in Q^+$.

[In order to allow certain symmetry in terms of some naturally generated mathematical sequences observed and derived from geometrical properties of right triangles of Fermat family; we have introduced the above form of triplet.]

6.2 Dual of a given triplet

In this section, we want to introduce E-Pythagorean triplets associated with a given Pythagorean triplet.

Let(a, b, h) with $a, b\ and\ h \in N$ be a primitive Pythagorean triplet. We recall certain properties that we have already established previously. They are as follows.

1. If $a \in N$ be an even integer then $b \in N$ is an odd integer but *not* a prime and h is in odd integer.
2. If $a \in N$ is an odd integer then obviously $b \in N$ is some even integer and h is an odd integer.

[Comment:- The first term 'a' of the primitive triplet (a, b, h) satisfying equation (1), In general $a \in \{3, 17, 99,\ldots\} = \{ a_n \mid a_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n],$ for $n \in N$ } then the second term **b** is a perfect square. The proof is given in Annexure-2.]

Let us introduce a new concept of deriving a set of E-Pythagorean triplets form a given Pythagorean triplets.

Let (a, b, h) be any given primitive triplet satisfying the conditions given in (1).

Let $a = l_1 \times m_1$ with $l_1 \le m_1$ and $b = l_2 \times m_2$ with $l_2 \le m_2$, where$l_1, m_1, l_2\ and\ m_2$ are some members of N.

For a, b and h; we have, $a^2 + b^2 = h^2$

Let $a = l_1 \times m_1, b = l_2 \times m_2$, h then we have the following triplets satisfying E-Pythagorean property;

These triplets shall be called 'dual' triplets to the given triplet(a, b, h).

Dual triplets for a given break-up are,

1) $a_1 = \frac{l_1}{l_2}, b_1 = \frac{m_2}{m_1},\ h_1 = \frac{h}{l_2 m_1}$
2) $a_2 = \frac{l_1}{m_2}, b_2 = \frac{l_2}{m_1},\ h_2 = \frac{h}{m_1 m_2}$
3) $a_3 = \frac{m_1}{l_2}, b_3 = \frac{m_2}{l_1},\ h_3 = \frac{h}{l_1 l_2}$
4) $a_4 = \frac{m_1}{m_2}, b_4 = \frac{l_2}{l_1},\ h_4 = \frac{h}{l_1 m_2}$

With each one of the triplets satisfying

$(a_i)^2 + (b_i)^2 = (h_i)^2$ for i = 1,2,3,4…

From the above fact we deduce that,

(1) Each factor pair of 'a' combining with all factors pair of 'b' makes 4 different triplets.(as shown above)

(2)If $a = l_1 \times m_1$ with $l_1 \le m_1,\ l_1, m_1 \in N$ has 'm' factor pairs and $b = l_2 \times m_2$with $l_2 \le m_2,\ l_2, m_2 \in N$ has 'n' factor-pairs then the total number of dual triplets will be 4mn.

Comments:

(a) If $a = 1 \times a$ and $b = 1 \times b$ then out of the four dual triplets one of them would be the original triplet.

1) $a_1 = \frac{1}{1}, b_1 = \frac{b}{a}, h_1 = \frac{h}{a}$
2) $a_2 = \frac{1}{b}, b_2 = \frac{1}{a}, h_2 = \frac{h}{ab}$
3) $a_3 = \frac{a}{b}, b_3 = \frac{1}{1}, h_3 = \frac{h}{b}$
4) $a_4 = \frac{a}{1}, b_4 = \frac{b}{1}, h_4 = \frac{h}{1}$ [This is an original triplet.]

All of them satisfying $(a_i)^2 + (b_i)^2 = (h_i)^2$ for i=1, 2, 3, 4

This implies that distinct dual triplets will be (4mn-1).

(b)If either **a** or **b** is perfect square then the number of distinct dual triplets will be (4mn-3).]

[For let $a = \alpha \times \alpha$ and $b = l \times m$ with (a, b, h) as a given primitive triplet, then we have only two distinct dual triplets; which are as follows.

$a = \alpha \times \alpha, l \times m$,   h

1) $a_1 = \frac{\alpha}{l}, b_1 = \frac{m}{\alpha}, \; h_1 = \frac{h}{l\alpha}$

2) $a_2 = \frac{\alpha}{m}, b_2 = \frac{l}{\alpha}, h_2 = \frac{h}{m\alpha}$]

(c) Combining the above comment (a) and (b); we conclude that the number of distinct dual triplets is either at the most (4mn-1) or at least (4mn-3).

e.g., for $a = 3$, b =4 and h = 5 ; for the triplet (3, 4 ,5)

We write, $a = 1 \times 3$, b = 1 x 4 and   h = 5.

The associated dual triplets are,

1) $a_1 = \frac{1}{1}, b_1 = \frac{4}{3}, h_1 = \frac{5}{3}$

2) $a_2 = \frac{1}{4}, b_2 = \frac{1}{3}, \; h_2 = \frac{5}{12}$

3) $a_3 = \frac{3}{4}, b_3 = \frac{1}{1}, \; h_3 = \frac{5}{4}$

4) $a_4 = \frac{3}{1}, b_4 = \frac{4}{1}, \; h_4 = \frac{5}{1}$ [original triplet (3,4,5)]

Also we have a $= 1 \times 3$, $b = 2 \times 2$, and h = 5, corresponding dual triplets are

5) $a_5 = \frac{1}{2}, b_5 = \frac{2}{3}, \; h_5 = \frac{5}{6}$

6) $a_6 = \frac{3}{2}, b_6 = \frac{2}{1}, \; h_6 = \frac{5}{2}$ which are exactly two as the term $b = 4$, is a perfect square in N.

Concluding the remarks, we have $2(= m)$ factor pairs for 'a' and $2(= n)$ factor pairs for 'b' then the total numbers of distinct, other than the original one, triplets are $4mn - 3 = 8 - 3 = 5$.

(Excluding the original triplet shown by 4 above) Each one is E-Pythagorean dual triplets satisfying

$(a_i)^2 + (b_i)^2 = (h_i)^2$for i=1, 2,3,4,5,6

Annexure – 1

Let (a, b, h) with a, b and h $\in N$be a primitive triplet satisfying the property given by (1). If a is an even positive integer of the form 2k, where k$\in N$, then we prove that b is not a prime.

As a property of (A), we have $a^2 + b^2 = h^2$

$\therefore b^2 = h^2 - a^2 = (2k + 1)^2 - (2m)^2$, where k, m $\in N$ and k > m and h is always an odd integer

$b^2 = 4 (k^2 - m^2) + 4k + 1$

$\therefore b^2 = 4[(k^2 - m^2) + k] + 1$, which is an odd integer.

Now to show that, b = $\sqrt{4[(k^2 - m^2) + k] + 1}$ is not a prime.

$$b = \sqrt{(2k + 1)^2 - (2m)^2}$$

$$= \sqrt{(2k + 2m + 1)(2k - 2m + 1)}$$

$$= \sqrt{(2(k + m) + 1))}\sqrt{(2(k - m) + 1))}$$

Let $k + m = \alpha$ and $k - m = \beta$, as, k > m

$= \sqrt{2\alpha + 1}\sqrt{2\beta + 1}$($\because 2\alpha + 1$ and $2\beta + 1$ both are odd and not equal to 1.)

As b is an odd integer, a product of two different odd integers and hence it cannot be a prime number.

Annexure – 2

Let (a, b, h) with b = α^2 be a primitive triplets satisfying the properties of (1), If we consider sequence $\{a_n\}$ = { 3,17, 99, 577,…….} then we get a sequence { b_n | n $\in N$ }such that each term of $b_n = \alpha^2$ with some $\alpha \in N$. For this sequence $\{a_n\}$, we have recurrence relations,

$a_n = 6a_{n-1} - a_{n-2}$, for all $n \geq 3$.   ----- (10)

To find n^{th} term of this sequence, consider $a_n = t^2$, $a_{n-1} = t$ and $a_{n-2} = 1$.

$\therefore t^2 = 6t - 1 \Longrightarrow t^2 - 6t + 1 = 0$

$\therefore t = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$

So let $t_1 = 3 + 2\sqrt{2}$ and $t_2 = 3 -- 2\sqrt{2}$ and we have $a_1 = 3$, $a_2 = 17$

Therefore, $a_n = \alpha\, t_1^{n-1} + \beta\, t_2^{n-1}$for n$\in N$.

For n = 1, $a_1 = \alpha + \beta = 3$

For n = 2, $a_2 = \alpha t_1 + \beta t_2 = 17$

$\therefore \alpha(3 + 2\sqrt{2}) + \beta(3 - 2\sqrt{2}) = 17$

$\therefore 3(\alpha + \beta) + 2\sqrt{2}(\alpha - \beta) = 17$

$\therefore 9 + 2\sqrt{2}(\alpha - \beta) = 17$

$\therefore 2\sqrt{2}(\alpha - \beta) = 8$

$\therefore (\alpha - \beta) = 2\sqrt{2}$

Solving them, we get $\alpha = \frac{(3 + 2\sqrt{2})}{2}$ and $\beta = \frac{(3 - 2\sqrt{2})}{2}$.

$\therefore a_n = \frac{(3 + 2\sqrt{2})}{2}(3 + 2\sqrt{2})^{n-1} + \frac{(3 - 2\sqrt{2})}{2}(3 - 2\sqrt{2})^{n-1}$

$\therefore a_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]$, for n $\in N$

e.g., for n=1, $a_1 = 3$ for n = 2, $a_2 = 17$, And so on….

Let t $\in$ T, then t = {(a, b, h) | (a, b, h) is a primitive triplet}, satisfying properties of equation (1)

If a is an odd positive integer then we have(a, b) =1 implies b is an even positive integer.

We have a very special interesting case. Let a – the first term $\in$ {3, 17, 99, 577, ….}

[the general term of the sequence is

$a_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]$, for n $\in N$]

In the case, where $\alpha \in \{a_n \mid a_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]$, for n $\in N$ }.

We want to establish that b is an (even) perfect square.

i.e., $\exists$ some $\alpha \ni b = \alpha^2$ and the triplet $(a, b = \alpha^2, h)$ is a Pythagorean primitive triplet.

We know that for some odd integer,

a, b = $\frac{a^2-1}{2}$, h = $\frac{a^2+1}{2}$satisfies $a^2 + b^2 = h^2$with all the required properties for (a, b, h) to be a primitive triplet.

For $a \in$ { a_n | $a_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n]$, for n $\in N$ }.

Let for some $n_1 \in N$. $a = \frac{1}{2}\left[\left(3 + 2\sqrt{2}\right)^{n_1} + \left(3 - 2\sqrt{2}\right)^{n_1}\right]$,

Now b = $\frac{a^2-1}{2}$ (as shown above)

$\therefore$ b = $\frac{\left[\frac{1}{2}\left\{\left(3+2\sqrt{2}\right)^{n_1}+\left(3-2\sqrt{2}\right)^{n_1}\right\}\right]^2-1}{2}$ ---- (11)

Now we want to show that 'b' is a perfect square; i.e., for some $\alpha \ni$ b = α^2.

i.e.,$\frac{1}{2}\left[\left(3 + 2\sqrt{2}\right)^{n_1} + \left(3 - 2\sqrt{2}\right)^{n_1}\right]^2 = 2\alpha^2 + 1$ is odd and perfect square.

$\therefore 2\alpha^2 + 1 = (2k + 1)^2$ for some k$\in N$.

$\therefore \alpha^2 = 2k(k+1)$ ----- (12)

and 2k (k+1) is a perfect square only for some k a member of {1,8,49,288,1681,…}

k = {1, 49, 1681…} U {8, 288, 9800…}

k = {1, 49, 1681…} U {8(1,36,1225….)}

k = $\{(a_n)^2 \mid a_n$ is a NSW sequence }U{ 8(1^2, 6^2, 35^2,….)}

k = $\{(a_n)^2 \mid a_n$ is a NSW sequence }U{ 8 $(J_n)^2$}, where J_n is a Jha. Sequence.

This gives the pattern of k for which $2k(k+1)$ is perfect square. Now we find the pattern of $b = \alpha^2$ for given $a \in \left\{a_n | a_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^n + \left(3-2\sqrt{2}\right)^n\right];\ \forall\, n \in N\right\}$. (This is derived in annexure-3).

Annexure -3

For given primitive Pythagorean triplet of the type (a, b, h) where $a \in \{3, 17, 99, 577, \dots\}$

A sequence of numbers with the general term,

$$a_n = \frac{1}{2}\left[\left(3+2\sqrt{2}\right)^n + \left(3-2\sqrt{2}\right)^n\right];\ \forall\, n \in N$$

The term 'b' satisfying the Pythagorean triplet is the corresponding terms of the set $B = \{4,144,4900, \dots\}$

i.e. $B = 4\{(1)^2, (6)^2, (35)^2, \dots\}$

By the reference of 'Jha Sequence'$\{1, 6, 35, 204, \dots\}$ For which the general term is given as

$$J_n = \left\{\frac{\sqrt{2}}{8}\left[\left(3+2\sqrt{2}\right)^n + \left(3-2\sqrt{2}\right)^n\right];\ \forall\, n \in N\right\} \quad \textbf{..(13)}$$

$\therefore$ Each term of the set B = $\{(2J_n)^2 \mid J_n\ is\ Jha\ sequence\}$

Conclusion:

The above note in four different tenets on Pythagorean family of triangles and its extended version to Fermat family is innovative in the sense that it has explored some basic concepts of abridging the two families. In addition, concepts related to duals to a given triplet has established a strong foundation stone in extending Pythagorean property to the set of positive rational numbers extending to completeness of the set of right triangles.

Vision:

We, inspired by the above work, have been able to deduce some additional properties of duals to a given triplet and finally are on the verge of establishing a rational field of triplets. This would follow in our forthcoming note in a near future

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