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A new approach for solving quadratic fractional programming Problems

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Abstract
A new method namely, *decomposition fractional separable method* based on linear programming (LP) problem is proposed for finding an optimal solution to a quadratic fractional programming (QFP) problem in which both the numerator and the denominator of the objective function can be factorized into two linear functions. The solution procedure of the proposed method is illustrated with the numerical example.

Keywords: Quadratic fractional programming problem, linear programming problem, decomposition fractional separable method.

1. Introduction
The quadratic fractional programming problems are the topic of great importance in nonlinear programming. They are useful in many fields such as production planning, financial and corporative planning, health care and hospital planning. In various applications of nonlinear programming, one often encounters the problem in which the ratio of given two quadratic functions is to be maximized or minimized. In this paper, we consider QFP problem in which the ratio of the quadratic function in the objective can be factorized into two linear functions. Several methods have been developed to solve such problems are proposed. Rashidul Hasan and Babul Hasan [5] have extended the usual simplex method for solving QFP problem with inequalities constraints. Archana Khurana and Arora [1] have studied for solving QFP when some of its constraints are homogeneous. Sharma and Jitendra Singh [7] have proposed a new approach which is based on the iterative procedure of simplex techniques for solving QFP problems. Nejmaddin and Maher [3, 4] have studied on solving QFP by using Wolfe's method [8] and a new modified simplex approach. Rashidul Hasan and Babul Hasan [6] have extended simplex method for solving QFP when some of its constraints are homogenous.

2. Preliminaries
The mathematical form of QFP problems is given as follows:

Maximize $$Z = \frac{C^T X + \frac{1}{2} X^T G_1 X}{D^T X + \frac{1}{2} X^T G_2 X}$$

Subject to $AX\ (\le, =, \ge)\ B$, $X \ge 0$

Where G_1 and G_2 are $(n \times n)$ matrix of coefficients with G_1, G_2 are symmetric matrices. All vectors are assumed to be column vectors unless transposed (T), where X is an n-dimensional vector of decision variables is, C and D are *n*-dimensional vector of constants, A is an $(m \times n)$ matrix and B is an *m*-dimensional vector of constants.

In this paper, we consider a QFP problem in which the quadratic function in the objective can be factorized into two linear functions. Such QFP problem can be represented as follows:

(P) Maximize $Z(X) = \frac{\left(C_1^T X + \alpha\right)\left(C_2^T X + \beta\right)}{\left(D_1^T X + \gamma\right)\left(D_2^T X + \lambda\right)}$

Subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$

Where X is an n-dimensional column vector of decision variables, C_1, C_2, D_1 and D_2 are n-dimensional column vector of constants, A is an $(m \times n)$ matrix and B is an m-dimensional row vector of constants and $\alpha, \beta, \delta, \gamma$ are scalars.

Now, we construct two QP problems namely (P_1) and (P_2) from the above QFP problem (P) as follows:

(P_1) Maximize
$Z_1(X) = \left(C_1^T X + \alpha\right)\left(C_2^T X + \beta\right) = Z_{11}(X) \bullet Z_{12}(X)$
Subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$
And

(P_2) Maximize
$Z_2(X) = \left(D_1^T X + \gamma\right)\left(D_2^T X + \lambda\right) = Z_{21}(X) \bullet Z_{22}(X)$
Subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$.

We need the following definitions that can be found in Jayalakshmi and Pandian [2].

Definition 1 Let $f_1(x)$ and $f_2(x)$ be two differentiable functions defined on $X \subset R^n$, an n-dimensional Euclidean space. The functions $f_1(x)$ and $f_2(x)$ are said to have the Gonzi property in $X \subset R^n$

if $(f_1(x) - f_1(u))(f_2(x) - f_2(u)) \leq 0$, for all $x, u \in X$.

Theorem 1 The product $f_1(x) f_2(x)$ of two linear functions $f_1(x)$ and $f_2(x)$ is concave if and only if the functions $f_1(x)$ and $f_2(x)$ has the Gonzi property.

Now, we assume that the functions $\left(C_1^T X + \alpha\right)$ and $\left(C_2^T X + \beta\right)$ as well as $\left(D_1^T X + \gamma\right)$ and $\left(D_2^T X + \lambda\right)$ satisfy the Gonzi property in the feasible set and the set of all feasible solutions to the problem (P_1) and (P_2) are non-empty and bounded. Thus, by the Theorem 1, it is concluded that the problem (P_1) and (P_2) is a concave non-linear programming problem with linear constraints. This implies that the optimal solution of the problem (P_1) and (P_2) exists and it occurs at an extreme point of the feasible region.

Now, in order to solve the above QP problem (P_1) and (P_2) by LP technique, we decompose it into two single objective LP problems namely (P_{11}); (P_{12}) and (P_{21}); (P_{22}) as given below:

(P_{11}): Maximize $Z_{11}(X) = \left(C_1^T X + \alpha\right)$ subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$	(P_{12}): Maximize $Z_{12}(X) = \left(C_2^T X + \beta\right)$ subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$
(P_{21}): Maximize $Z_{21}(X) = \left(D_1^T X + \gamma\right)$ subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$.	(P_{22}): Maximize $Z_{22}(X) = \left(D_2^T X + \lambda\right)$ subject to $AX \ (\leq, =, \geq)\ B$, $X \geq 0$.

3. Decomposition Fractional Separable Method
Now, we proposed a new method namely, *decomposition fractional separable method* for finding an optimal solution to the given QFP problem with the help of LP technique.

The proposed method proceeds as follows
Step 1: Construct two QP problems namely (P_1) and (P_2) from the given QFP problem (P).

Step 2: Decompose the QP problem (P_1) and (P_2) into two single objective LP problems namely (P_{11}); (P_{12}) and (P_{21}); (P_{22}).

Step 3: Solve the problem (P_{11}) and (P_{21}) separately, with the help of LP technique. Let the optimal solution to the problem (P_{11}) be X_{11° and Max. $Z_{11}(X) = Z_{11}(X_{11^\circ})$ And the optimal solution to the problem (P_{21}) be X_{21° and Max. $Z_{21}(X) = Z_{21}(X_{21^\circ})$.

Step 4: Use the optimal table of the problem (P_{11}) and (P_{21}) as an initial table for the problem (P_{12}) and (P_{22}), continue to find a sequence of improved basic feasible solutions $\{X_n\}$ to the problem (P_{12}) and (P_{22}), also find the value of objective function of $Z_1(X)$ and $Z_2(X)$ at each of the improved basic feasible solution by LP technique.

Step 5: (a) If $Z_r(X_k) \leq Z_r(X_{k+1})$, $(r = 1, 2)$, for all $k = 0,1,2,\ldots, n-1$ and $Z_r(X_n) \geq Z_r(X_{n+1})$, $(r = 1, 2)$, for some n, stop the computation process and then, go to Step6.

Step 5: (b) If $Z_r(X_k) \leq Z_r(X_{k+1})$, for all $k = 0,1,2,\ldots, n$ and X_{n+1} is an optimal solution to the QP problem for some n, stop the computation process and then, go to Step 7.

Step 6: X_n is an optimal solution to the QP problem.

Step 7: X_{n+1} is an optimal solution to the QP problem.

Step 8: Collect all the feasible solution from the iterations of the problem (P_{12}) and (P_{22}), also compute the value of the objective function of QFP problem (P), $(i.e)$ $Z(X)$, then maximum value of $Z(X)$ among all the feasible solution is the optimal solution for the QFP problem (P).

The proposed method for solving the QFP problem is illustrated by the following examples.

Example 1 Consider the following QFP problem:

(P) Maximize $Z = \dfrac{(x_2+1)\,(x_1+x_2+3)}{(x_1+4)\,(x_1+x_2+2)}$

Subject to

$-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7;\ x_1, x_2 \ge 0.$

Using Step 1, construct two QP problems namely (P_1) and (P_2) from the above QFP problem (P) as:

$(P1)$: Maximize $Z_1(X) = Z_{11}(X)Z_{12}(X)$ subject to $-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7; x_1, x_2 \ge 0$, Where $Z_{11}(X) = x_2 + 1$ and $Z_{12}(X) = x_1 + x_2 + 3$	$(P2)$: Maximize $Z_2(X) = Z_{21}(X)Z_{22}(X)$ subject to $-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7$; $x_1, x_2 \ge 0$, Where $Z_{21}(X) = x_1 + 4$ and $Z_{22}(X) = x_1 + x_2 + 2$

Now, using Step 2, the above QP problem $(P1)$ and $(P2)$ decomposes into two single objective LP problems as given below:

(P_{11}): Maximize $Z_{11}(X) = x_2 + 1$ Subject to $-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7; x_1, x_2 \ge 0$.	(P_{12}): Maximize $Z_{12}(X) = x_1 + x_2 + 3$ Subject to $-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7; x_1, x_2 \ge 0$.
(P_{21}): Maximize $Z_{21}(X) = x_1 + 4$ Subject to $-x_1 + x_2 \le 1;\ x_1 + 2x_2 \le 7;\ x_1, x_2 \ge 0$.	(P_{22}): Maximize $Z_{22}(X) = x_1 + x_2 + 2$ Subject to $-x_1 + x_2 \le 1; x_1 + 2x_2 \le 7;\ x_1, x_2 \ge 0$.

Using Step 3, the optimal table for the problem (P_{11}) by simplex method is:

.	C	0	1	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
1	x_2	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{8}{3}$	
0	x_1	1	0	$\frac{-2}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	
$Z_j - C_j$		0	0	$\frac{1}{3}$	$\frac{1}{3}$	$Z_{11}(X) = \frac{11}{3}$	$Z_1(X) = \frac{242}{9}$

Therefore, the optimal solution to the problem (P_{11}) is $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}$, max. $Z_{11}(X) = \frac{11}{3}$ And the value of $Z_1(X) = \frac{242}{9}$

Now, by Step 4, the initial simplex table to the problem (P_{12}) is given below:

Initial table

	C	1	1	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
1	x_2	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{8}{3}$	
1	x_1	1	0	$\frac{-2}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	
$Z_j - C_j$		0	0	$\frac{-1}{3}$	0	$Z_{12}(X) = \frac{22}{3}$	$Z_1(X_\circ) = \frac{242}{9}$

Entering variable is s_1 and leaving variable is x_2.

IST iteration table

	C	1	1	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
0	s_1	0	3	1	1	8	
1	x_1	1	2	0	1	7	
$Z_j - C_j$		0	1	0	1	$Z_{12}(X)=10$	$Z_1(X_1)=10$

Since $Z_1(X_\circ) > Z_1(X_1)$ and by Step 5(a) of the proposed method, the optimal solution to the QP problem $(P1)$ is $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}$ and max. $Z_1(X) = \frac{242}{9}$.

Similarly, using Step 3, the optimal table for the problem (P_{21}) by simplex method is:

	C	0	1	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
0	s_1	0	3	1	1	8	
1	x_1	1	2	0	1	7	
$Z_j - C_j$		0	0	$\frac{1}{3}$	$\frac{1}{3}$	$Z_{21}(X)=7$	$Z_2(X)=99$

Therefore, the optimal solution to the problem (P_{21}) is $x_1 = 7$, $x_2 = 0$, max. $Z_{21}(X) = 7$ And the value of $Z_2(X) = 99$.

Now, by Step 4, the initial simplex table to the problem (P_{22}) is given below:

Initial table

	C	1	1	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
0	s_1	0	3	1	1	8	
1	x_1	1	2	0	1	7	
$Z_j - C_j$		0	0	$\frac{1}{3}$	$\frac{1}{3}$	$Z_{22}(X)=9$	$Z_2(X)=99$

Since the initial iteration table is optimal and by Step 5(b) of the proposed method, the optimal solution to the QP problem $(P2)$ is $x_1 = 7$, $x_2 = 0$ and max. $Z_2(X) = 99$.

To find the optimal solution for the given QFP problem (P), we collect all the feasible solution from the iterations of the problem (P_{12}) and (P_{22}) which are listed below:

Solutions	**Feasible Solutions** (x_1, x_2, s_1, s_2)	**Objective Value of** $Z(X)$
1	$X_0 = \left(\frac{5}{3}, \frac{8}{3}, 0, 0\right)$	$\frac{242}{323}$
2	$X_1 = (7, 0, 0, 0)$	$\frac{10}{99}$

Now, from the above table, the optimal solution for the given QFP problem (P) is $x_1 = \frac{5}{3}, x_2 = \frac{8}{3}$ and Maximum $Z(X) = \frac{242}{323}$.

Remark: The solution of the Example 1 is same as Nejmaddin and Maher [3], but they defined a new modified simplex method to solve QFP problem, where else the solution is obtained only by using simplex method (LP technique).

4. Conclusion

The decomposition fractional separable method is proposed to solve QFP problems in which the ratio of the quadratic function in the objective can be factorized into two linear functions. This method depends on transforming the given QFP problem in QP problem, which is then decomposed in LP problem. Using the optimal solution of the QP problem which occurs at an extreme point of the feasible region, the optimal solution to the QFP problem is obtained. Since the proposed method is based only on LP technique, it is very easy to compute and to apply. Also, we can solve such QFP problems using the existing LP solvers. Further, the present work can be extended to integer and fuzzy QFP problems.

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