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Regular number of line graph of a graph

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Abstract
For any (p, q) graph G, a line graph L(G) is obtained from G by taking each edge as a vertex in L(G). The regular number of the L(G) is the minimum number of subsets into which the edge set of L(G) should be partitioned so that the sub graph induced by each subset is regular and is denoted by r_L(G). In this paper some results on regular number of r_L(G) were obtained and expressed in terms of elements of G.

Keywords: Regular number, Line graph, Domination number, Total domination.

1. Introduction

All graphs considered here are simple, finite, non-trivial. As usual p and q denote the number of vertices and edges of a graph G and, the maximum degree of a vertex in G is denoted by Δ(G). A vertex v is called a cut vertex if removing it from G increases the number of components of G. A graph G is called trivial if it has no edges. The maximum distance between any two vertices in G is called the diameter, denoted by diam (G). A tree is called a binary tree if it has one vertex of degree 2 and each of the remaining vertices is of degree 1 or 3. The path and tree numbers were introduced by Stanton James and Cown in [10]. Any undefined term in this paper may be found in [2]. Let G = (V, E) be a graph. A set $D' \subseteq$ V is said to be a dominating set of G, if every vertex in (V – D') is adjacent to some vertex in D'. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by γ(G). A dominating set is said to be total dominating set of G, if N(D') = V or equivalently, if for every $v \in V$, there exists a vertex u ∈ D' , u $\neq v$, such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G. Domination related parameters are now well studied in graph theory. The total domination $\gamma_t(G)$ were studied by M.H. Muddebihal, Srinivasa. G, and A.R. Sedamkar in [6]. A dominating set D of L(G) is a regular total dominating set (RTDS) if the induced sub graph < D > has no isolated vertices and deg(v) = 1, ∀ $v \in D$. The regular total domination number $\gamma_{rt}(L(G))$ is the minimum cardinality of a regular total dominating set. The regular total domination in line graphs were studied by M.H. Muddebihal, U.A. Panfarosh and Anil. R. Sedamkar in [7]. A total dominating set in a graph were studied by M.A. Henning and A. Yeo in [4]. Total domination in graphs were studied by E.J. Cockayne, R.M. Dawes, and S.T. Hedetniemi in [1]. Total domination and total domination subdivision numbers of graphs were studied by O. Favaron, H. Karami and S.M. Sheikholeslami in [9]. Recent results on total domination in graphs were studied by M. A. Henning in [3]. On matching and total domination in graphs, were studied by M. A. Henning, L. Kang, E. Shan and A. Yeo in [5]. The total {k}-domination number of Cartesian products of graphs were studied by N. Li and X. Hou in [8].

2. Results

The following result is obvious, hence we omit its proof.

Theorem 1: For any regular graph G, $r_L(G)$ = 1.
Next, we obtain the regular number of a path.

Theorem 2: For any path P_p , $r_L(P_p)$ = 2.

Proof: Let P_p : $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, . . . , $e_{p-2} = v_{p-2}v_{p-1}$, $e_{p-1} = v_{p-1}v_p$ be a path. Now, we have, L(P_p) : e_1e_2 , e_2e_3 , e_3e_4 , . . . , $e_{p-2}e_{p-1}$ be the edge set of L(P_p). Let F_1 = {e_1e_2 , e_3e_4 , e_5e_6 , . . . , $e_{p-3}e_{p-2}$ } and F_2 = { e_2e_3 , e_4e_5 , e_6e_7 , . . . , $e_{p-2}e_{p-1}$ } be the minimum regular partition of L(P_p). Hence, $r_L(P_p)$ = │F_1 , F_2} │, clearly ,$r_L(P_p)$ = 2.

In the following theorem we establish the regular number of line graph of a binary tree.

Theorem 3: For any non-trivial binary tree T, $r_L(T) \leq$ Δ(T). Where Δ(T) is the maximum degree of T.

Proof: Let T be a non-trivial binary tree. Suppose, a binary tree T has p ≥ 5 vertices, then there exists only $F_1$and F_2 partitions of L(T). Thus, Δ(T) > r_L(T). Suppose a binary tree T has p ≥ 7 vertices, then the edges incident to the vertex of degree 2 gives a bridge in L(T) and remaining blocks of L(T) are K_3. Now, the set of K_3 's are partitioned into two sets which are adjacent to each other and another set contains only a bridge of L(T). Hence, in any partition of L(T), we have F_1 , F_2 , and F_3. Since, F = {F_1, F_2, F_3} is the minimum regular partition of r_L(T).

Thus, r_L(T) ≤ │{ F_1 , F_2 , F_3 }│ clearly, r_L(T) ≤ 3 which gives r_L(T) ≤ Δ(T).

In the next result we obtain the regular number of line graph of a complete bipartite graph.

Theorem 4: For any complete bipartite graph $K_{m,n}$, $r_L(K_{m,n})$ = 1.

Proof: Let G = $K_{m,n}$ be a complete bipartite graph. Then for any $K_{m,n}$, if m > n. Then L[$K_{m,n}$] = m-regular which gives only one partition. Hence, $r_L[K_{m,n}]$ = 1. For a $K_{m,n}$, if n > m. Then L[$K_{m,n}$] = n-regular. Thus,$r_L[K_{m,n}]$ = 1. Further for a $K_{m,n}$, if m = n , then L[$K_{m,n}$] = m or n-regular. On all partitions of L[$K_{m,n}$], we have $r_L(K_{m,n})$ = 1. Hence, for any $K_{m,n}$, $r_L[K_{m,n}]$ = 1.

Next, we obtain the result of regular number of line graph of a complete graph.

Theorem 5: For any complete graph K_p , then $r_L(K_p)$ = 1 .

Proof: Let v_1 , v_2 , v_3 , . . . , v_p be the vertices of K_p and each vertex is of degree p-1. Let e_1 , e_2 , e_3 , . . . , $e_{\frac{p(p-1)}{2}}$ be the edges of K_p such that $e_1 = v_1v_2$, $e_2 = v_1v_3$, $e_3 = v_1v_4$, . . . , $e_{\frac{p(p-1)}{2}} = v_{p-1}v_p$

Now, in L(K_p) the edge set corresponds to the vertex set. Let e_1 , e_2 , e_3 , . . . , $e_{\frac{p(p-1)}{2}}$ be the vertices of L(K_p) and every vertex is regular of degree 2(p-2). Hence, the regular partition is one. Thus, $r_L(K_p)$ = 1. Further the above proof can be stated in terms of regularity of G such as; since K_p is a regular graph with p-1 regular. Also the degree of each edge is p-regular. In, L(G) E(G) = V[L(G)], hence degree of each vertex in L(G) = p. Clearly L(K_p) is p-regular. Hence, $r_L(K_p)$ = 1.

Now, we obtain the exact value of $r_L(T)$ and the cut vertices.

Theorem 6: For any tree with n ≥ 2cut vertices with same degree, $r_L(T)$ = 2.

Proof: Let T be a non-trivial tree with n-cut vertices with same degree. In L(T), each block is complete with same regular. Since each cut vertex of L(T) is incident with exactly two blocks. Then each block belongs to different partitions of L(T) which are F_1 and F_2. Then, $F = \{ F_1 , F_2 \}$ such that │F_1 , F_2 │=$r_L(T) = 2$.

Next, we obtain the result of regular number of line graph of a wheel.

Theorem 7: For any wheel W_p, with p ≥ 5 vertices, $r_L(W_p)$ = 2.

Proof: Let v_1 , v_2 , v_3 , . . . , v_p be the vertices of W_p such that degv_i = 3 for $1 \leq i \leq$ p – 1 and degv_p = p – 1. Let e_1 , e_2 , e_3 , . . . , e_{p-1}, e_1' , e_2' , e_3' , . . . , e_{p-1}' be the edges of W_p such that $e_i = v_iv_{i+1}$ for 1 ≤ i ≤ p – 2 , $e_{p-1} = v_1v_{p-1}$ and $e_i' = v_iv_p$ for $1 \leq i \leq p-1$. Now, in L(W_p) e_1 , e_2 , e_3 , . . . , e_{p-1}, e_1' , e_2' , e_3' , . . . , e_{p-1}' be the vertices of L(W_p) which corresponds to the edges of W_p in G. Let F_1 = { $e_1e_2e_3$, . . . , $e_{p-2}e_{p-1}e_1'e_2'e_3'$, . . . , $e_{p-2}'e_{p-1}'$ } in which deg(e_i) = 4 = deg(e_j') ∀ e_i , e_j' ∈ F_1 and < F_1 > is 4-regular. Similarly for F_2 = { $e_1'e_3'e_5'$, . . . , $e_{p-1}'e_2'e_4'e_6'$, . . . , $e_{p-2}'e_1'e_4'e_7'e_{10}'$, . . . , $e_{p-6}'e_{p-3}'e_1'$ } ∀ e_k' ∈ F_2 , deg(e_k') = p – 4 , k = 1 , 2 , 3 , . . . ,p – 1 , is the another partition such that < F_2 > is (p – 4) regular graph. Thus, $r_L(W_P)$ = │{ F_1 , F_2 } │ = 2.

In the next result we establish the relationship between $r_L(T)$ *and* n where n is n-distinct cut vertices.

Theorem 8: For any non-trivial tree T, with n-distinct cut vertices, then $r_L(T) = n$.

Proof: Suppose G = T has unique cut vertex incident with m number of edges in L(G), the sub graph < L(G) > = K_m which is m-1 regular. Now, $r_L(T) = n = 1$. Further, assume $n \geq 2$ and each n is distinct. Let T has {2} , {3} , . . . , {n} number of cut vertices and each set has let v_1 , v_2 , v_3, . . . , v_n be the number of cut vertices in T ; deg(v_1) ≠ deg(v_2) ≠ deg(v_3) ≠ , . . . , ≠ deg(v_n) and ∀ v_i ∈ T has degree at least 2. Since each v_i , $1 \leq i \leq n$ are distinct. Let { e_1 , e_2 , e_3 , . . . , e_p } be the number of edges in T, which are incident to ∀ v_i $1 \leq i \leq n$. Then in L(T), the number of edges incident to each v_i ∈ G generates a complete block in L(T). In L(T) each block is of different regular and is complete. Since T has n- distinct cut vertices and each block is of different regular, then each block belongs to different partition of L(T) such that there exists F_1 , F_2 , F_3 , . . . , F_n partitions of L(T). Hence, $r_L(T)$ = │ { F_1 , F_2 , . . . , F_n} │ which gives $r_L(T)$ = n.

Next, we developed the result which gives the relationship between $r(G)$ *and* $diam[L(G)]$.

Theorem 9: For any graph G, $r(G) \leq q - diam[L(G)] + 1$.

Proof: Let P_n : { v_1v_2 , v_2v_3 , v_3v_4 , . . . , $v_{p-2}v_{p-1}$, $v_{p-1}v_p$ } be a path on diam[L(G)] + 2 vertices. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_5$, . . ,

$e_{p-4} = v_{p-4}v_{p-3}$, $e_{p-3} = v_{p-3}v_{p-2}$, $e_{p-2} = v_{p-2}\, v_{p-1}$, $e_{p-1} = v_{p-1}v_p$ be the edges of P_p. In $L(P_p)$ the set $\{ e_1 , e_2 , e_3 , \ldots , e_{p-1} \} \in V[L(P_p)]$. Then clearly $F_1 = \{ e_1e_2 , e_3e_4 , e_5e_6 , \ldots , e_{p-3}e_{p-2} \}$ and $F_2 = \{ e_2e_3 , e_4e_5 , e_6e_7 , \ldots , e_{p-2}e_{p-1} \}$ is the minimum regular partition of $L(P_p)$. Let $e_1 , e_2 , e_3 , \ldots , e_{q-(p-2)}$ be the edges which do not lie on P_p . Then, $F = \{\{e_1\}, \{e_2\}, \{e_3\}, \ldots, \{e_{q-(p-2)}\}, F_1 , F_2 \}$ is the regular partition of G.

Thus, $r(G) \leq |F|$

$r(G) \leq q - (p-2) + 2$

$r(G) \leq q - p + 4$

$r(G) \leq q - (p-3) + 1$

$r(G) \leq q - diam[L(G)] + 1$.

Now, we obtain the another result relationship between $r_L(G)$ and $diam(G)$.

Theorem 10: For any graph G, then $r_L(G) \leq q - diam(G) + 2$.

Proof: Let $P_p : v_1v_2 , v_2v_3 , v_3v_4 , \ldots , v_{p-1}v_p$ be a path on $diam(G) + 1$. Let $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, . . , $e_{p-4} = v_{p-4}v_{p-3}$, $e_{p-3} = v_{p-3}v_{p-2}$, $e_{p-2} = v_{p-2}v_{p-1}$, $e_{p-1} = v_{p-1}v_p$ be the edges of P_p . In $L(P_p)$, $\{ e_1 , e_2 , e_3 , \ldots , e_{p-1} \} \in V[L(G)]$. Then clearly $F_1 = \{ e_1e_2 , e_3e_4 , e_5e_6 , \ldots , e_{p-3}e_{p-2} \}$ and $F_2 = \{e_2e_3 , e_4e_5 , e_6e_7 , \ldots , e_{p-2}e_{p-1} \}$ is the minimum regular partition of $L(P_p)$. Let $e_1 , e_2 , e_3 , \ldots , e_{q-(p-1)}$ be the edges which do not lie on P_p . Then $F = \{ \{e_1\}, \{e_2\}, \{e_3\}, \ldots, \{e_{q-(p-1)}\}, F_1 , F_2 \}$ is the regular partition of L(G).

Thus, $r_L(G) \leq |F|$

$r_L(G) \leq q - (p-1) + 2$

$r_L(G) \leq q - diam(G) + 2$.

In the following theorem we establish the relationship between r_L(T) and $\gamma_t(T)$.

Theorem 11: For any non-trivial tree T with distinct degrees cut vertices, then r_L(T) = $\gamma_t(T)$.

Proof: Suppose tree T has n cut vertices which are of distinct degrees. Then L(T) has a set of blocks $\{ B_1 , B_2 , B_3 , \ldots , B_n \}$ in which each $< B_i >$ is complete $1 \leq i \leq n$ and $\forall B_i \in L(T)$ has different regular. Hence by Theorem 8 $r_L(T) = n$. Suppose every cut vertex of T has distinct degrees. Then clearly each cut vertex is adjacent to at least one end vertex. Let $C = \{C_1 , C_2 , C_3 , \ldots , C_n \}$ be the set of cut vertices in T such that $d(C_n) > d(C_{n-1}) > \ldots > d(C_1)$ and $\forall C_i \in C$ $1 \leq i \leq n$ has at least one end vertex. Then $d(C_i) > 2$ $1 \leq i \leq n$. Suppose D is a dominating set. Then D = C such that $< C >$ is connected, hence D is a minimal total dominating set such that $|$D$|$ = $\gamma_t(T)$. If there exists a vertex $v \in V(T)$ such that v has no end vertex. Clearly $d(v) = 2$. Then $C - \{v\}$ has a isolates as v. Hence to get $\gamma_t(T)$ set we require $\cup \{v\}$, which gives $< C \cup \{v\} >$ without isolates. Hence T has n distinct degree cut vertices gives $\gamma_t(T) = n$. Again from the above $\gamma_t(T) = r_L(T)$.

In the next result we developed a relationship between $r_L(T)$ and $\gamma(T)$.

Theorem 12: For any non-trivial tree T, if every cut vertex of T is γ-set, then $r_L(T) \leq \gamma(T)$.

Proof: For any tree T, let $D = \{ v_1 , v_2 , v_3 , \ldots , v_n \}$ be the set of non end vertices which are cut vertices in T. Suppose every vertex of D is adjacent to at least one end vertex. Then D is a γ-set of T. Suppose $\deg(v_1) > \deg(v_2) > \deg(v_3) >$ _ _ _ $> \deg(v_n)$. Then in L(G) $\forall v_i \in D$ gives n-distinct regular partition. Hence $|D| = n = r_L(T)$ which gives $r_L(T) = \gamma(T)$. For inequality suppose $D_1 = \{ v_1 , v_2 , v_3 , \ldots , v_i \}$ such that $D_1 \subset D$ and $\forall v_i \in D_1$ are not in γ-set of T. We consider the following cases.

Case 1. Assume $\forall v_i \in D_1$ are not adjacent to each other. Then, in L(T) the edges incident to each v_i forms a regular block in L(T) which are not adjacent. Hence there blocks which are with same regular and are not adjacent to each other given only one partition. Remaining blocks in L(T) are in different regular classes. Hence, $r_L(T) \leq \gamma(T)$.

Case 2. Assume $\forall v_i \in D_1$ are adjacent. Then in L(T) the edges incident to these vertices form same regular and hence they are in different regular partition. Thus $|D_1| > \gamma(T)$ which gives $r_L(T) > \gamma(T)$. Hence in all these we have $r_L(T) \leq \gamma(T)$.

3 Conclusion

We established the regular number of line graph of some standard graphs by replacing the each edge by a vertex. Also many results established are sharp.

4. References

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