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# On Ternary Quadratic Equation $4x^2 + 5y^2 = 21z^2$

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**Abstract**
The Ternary Quadratic Diophantine Equation with 3 unknowns given by $4x^2 + 5y^2 = 21z^2$ is analyzed for its patterns of non-zero distinct integral solutions. Afew interesting relations between the solutions and special polygonal numbers are exhibited.

**Keywords:** Ternary Quadratic, Integral Solutions, Special Polygonal number, Centered Polygonal number.

## Introduction

### Synopsis

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety [1-5]. For an extensive review of various problems,one may refer [6-20]. This communication concerns with yet another interesting ternary quadratic equation $4x^2 + 5y^2 = 21z^2$ for determining its infinitely many non-zero integral solutions.Also a few interesting relations among the solutions have been presented.

### Notations Used

- $T_{m,n}$ -Polygonal number of rank n with size m.
- $CP_{m,n}$ - Centered Polygonal number of rank n with size m.
- $Pr_n$ - Pronic number of rank n.
- $S_n$ - Star number of rank n.
- $SO_n$ -Stella Octangular number of rank n.
- $Obl_n$ -Oblong number of rank n.
- $OH_n$ - Octahedral number of rank n.
- $Tet_n$ -Tetrahedral number of rank n.
- $Pt_n$ -Pentatope number of rank n.
- $F_{4,n-4}$ - Four dimensional figurative number of rank n whose generating polygonal is a square.
- $PP_n$ - Pentagonal Pyramidal number of rank n.

### Methodof Analysis

The Ternary Quadratic Equation to besolvedin integers is

$$4x^2 + 5y^2 = 21z^2 \qquad (1)$$

It is noted that (1) can be satisfied by the following triples of integers (-502a, 3344a,-276a), (-416a, 331a,-243a), (-446A, 328A,-252A), (492A, 489A,-321A), (-734T, 445T,-387T).
However, we have different pattern of solutions of (1) which are illustrated below:

### Pattern –I

Equation (1) is equivalent to the system of double equation

$$2Bx - Ay - (A + 4B)z = 0$$
$$2Ax + 5By + (4A - 5B)z = 0$$

Hence the corresponding solution of (1) given by

$$x = x(A, B) = -4A^2 + 10AB + 20B^2$$

$y = y(A, B) = -2A^2 - 16AB + 10B^2$
$z = z(A, B) = 2A^2 + 10B^2$

A few interesting properties observed are as follows
1.$x(A, A+1) - 2y(A, A+1) - 42Pr_A \equiv 0$
2.$y(A, 2A^2+1) + z(A, 2A^2+1) - x(A, 2A^2+1) + 78OH_n - T_{10,A} \equiv 0 (mod 3)$
3.$y(A(A+1), B) + z(A(A+1), B) - T_{42,B} + 32CT_{m,n} \equiv 32 \ (mod\ 19)$
4.$x(A, 1) + S_A - T_{6,A} \equiv 19 \ (mod\ 5)$

**Pattern –II**
The substitution of linear transformations $(u \neq v \neq 0)$
$x = u + 5v, \ y = u - 4v$, and $z = 3z$ (2)
In (1) leads to $u^2 + 20v^2 = 21z^2$ (3)
Assume $z = a^2 + 20b^2$ (4)

Where a and b are non-zero distinct integers.
21 can be written as

$$21 = (1 + i2\sqrt{5})(1 - i2\sqrt{5}) \quad (5)$$

Using (4) and (5) in (3) and employing method of factorization, define

$$u + i2\sqrt{5}\,v = (1 + i2\sqrt{5})(a + i2\sqrt{5}b)^2 \quad (6)$$

Equating real and imaginary parts in (6) and using (2), the values of x and y satisfies (1) are given by

$$x = x(a, b) = 6a^2 - 30ab - 120b^2$$
$$y = y(a, b) = -3a^2 - 48ab + 60b^2$$
$$z = z(a, b) = 3a^2 + 60b^2$$

**Properties**
1.$x(a, a(a+1)) + y(a, a(a+1)) + z(a, a(a+1)) - T_{14,a} + 156PP_a \equiv 0 \ (mod\ 5)$
2.$x(a, (a+1)(a+2)(a+3)) + 2y(a, (a+1)(a+2)(a+3)) + 3024Pt_a \equiv 0$
3.$y(a, 7a^2-4) + z(a, 7a^2-4) - T_{242,b} + 144CP_a^{14} \equiv 0 (mod\ 119)$
4.$x(a, a+1) + 2z(a, a+1) - T_{26,a} + 30Pr_a \equiv 0 (mod\ 11)$

**Pattern- III**
Instead of (5), one may write 21 as

$$21 = \frac{(8+i2\sqrt{5})(8-i2\sqrt{5})}{4}$$

Following the procedure similar to the above, the non-zero distinct integer values of x, y and z satisfying (1) are given by

$$x = x(A, B) = 52A^2 + 160AB - 1040B^2$$
$$y = y(A, B) = 16A^2 - 416AB - 320B^2$$
$$z = z(A, B) = 48A^2 + 960B^2$$

**Properties**
1.$3y(A, A(A+1)) + z(A, A(A+1)) + 2496PP_A - T_{194,A} \equiv 0 \ (mod\ 95)$
2.$z(B+1, B) - 3y(B+1, B) - 1248obl_B - T_{3842,B} \equiv 0 (mod 1919)$
3.$y(B^2+1, B) - y(B^2-1, B) - 85Pr_B + T_{44,B} \equiv 0 (mod\ 937)$
4. $z(A, 1) - 60obl_B + T_{26,A} \equiv 960 (mod\ 71)$

**Pattern-IV**
Introducing the linear transformation
$v = X + 21T, \ z = X + 20T, \ u = U$ (3) Leads to
$X^2 = U^2 + 420T^2$

Which is satisfied by

$$X = 420m^2 + n^2$$
$$T = 2mn$$
$$U = 420m^2 - n^2$$

Thus the corresponding solutions of (1) are

$$x = x(m, n) = 2520m^2 + 4n^2 + 210mn$$
$$y = y(m, n) = -1260m^2 - 5n^2 - 168mn$$

$z = z(m, n) = 1260m^2 + 3n^2 + 120mn$

**Properties**

1.$y(2n^2 + 1, n) + z(2n^2 + 1, n) - 4Pr_n + T_{14,n} + 144OH_n \equiv 0 (\text{mod } 9)$
2.$x(n(n+1), n) - 2z(n(n+1), n) + 60PP_n + 2Pr_n \equiv 0 (\text{mod } 2)$
3.$x(n^2, n) + y(n^2, n) - z(n^2, n) - 78CP_n^6 + T_{10,n} \equiv 0 (\text{mod } 3)$
4.$3y(m, m+1) + 5z(m, m+1) - 2520obl_m - 96Pr_n \equiv 0 (\text{mod } 2520)$

**Conclusion**

To conclude, one may search for other patterns of solutions and their corresponding properties.

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