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A fixed point theorem for a self-map on a fuzzy metric space and an application

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Abstract
The purpose of this paper is to obtain some results on existence of fixed points for contractive mappings in fuzzy metric spaces using control function. We prove our results on fuzzy metric spaces in the sense of Grabiec. Our results mainly generalize and extend the result of Gupta *et al.* [9]. As an application, a theorem of integral type contraction is given in support of our result.

Keywords: fixed point theorem; fuzzy metric space; L-integrable

Introduction

The concept of fuzzy set was introduced by Zadeh (1965) [17] and since then it has been developed extensively by many authors in different fields. The role of fuzzy topology in logic programming and algorithm has been recognized and applied on various programs to find more accurate results. In the last 50 years, this theory has wide range of applications in diverse areas. The strong points about fuzzy mathematics are its fruitful applications, especially outside mathematics, such as in quantum particle physics studied by El Naschie (2004) [2].
To use this concept in topology and analysis, Kramosil and Michalek (1975) [10] have introduced the concept of fuzzy metric space using the concept of continuous triangular norm defined by Schweizer (1960) [13]. Most recently, Gregori, Morillas, and Sapena (2011) [5] utilized the concept of fuzzy metric spaces to color image processing and also studied several interesting examples of fuzzy metrics in the sense of George and Veeramani (1994) [3].

Definition 1.1 (Schwizer 1960) [13] A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a **continuous triangular norm (t-norm)** if for all $a, b, c, e \in [0,1]$ the following conditions are satisfied:
(i) $*$ is commutative and associative,
(ii) $a * 1 = a$,
(iii) $*$ is continuous, and
(iv) $a * b \leq c * e$ whenever $a \leq c$ and $b \leq e$.
A fuzzy metric space in the sense of Kramosil and Michalek (1975) is defined as follows:

Definition 1.2 (Kramosil and Michalek 1975) [10] the triplet $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary non empty set, $*$ is a continuous t-norm, and M is fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:
(i) $M(x,y,0) = 0$
(ii) $M(x,y,t) = 1 \;\forall\, t > 0$ iff $x = y$
(iii) $M(x,y,t) = M(y,x,t)$
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s) \;\forall x,y,z \in X$ and $t, s > 0$
(v) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous, and
(vi) $\lim_{t\to\infty} M(x,y,t) = 1 \;\forall\, x, y \in X$.
The triplet $M(x,y,t)$ can be taken as the degree of nearness between x and y with respect to $t \geq 0$.
The following Lemma can be easily established

Lemma 1.3 For every $x, y \in X$, the mapping $M(x, y, \cdot)$ is non-decreasing on $(0, \infty)$.
Grabiec (1988) [4] extended the fixed point theorem of Banach (1922) [1] to fuzzy metric space in the sense of Kramosil and Michalek (1975) [10].
Now we give some important definitions and lemmas that are used in the sequel.

Definition 1.4 (Grabiec 1988) [4] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_{n,} x, t) = 1\ \forall\ t > 0$.

Definition 1.5 (Grabiec 1988) [4] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called a
Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p,} x_n, t) = 1\ \forall\ t > 0$ *and each positive integer* p.

Definition 1.6 (Grabiec 1988) [4] A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X converges in X.
We assume that the function $M(x, y, t)$ is continuous in the variables x *and* y.
Using (vi), the following lemma can be established.

Lemma 1.7 If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)\ \forall x, y \in X$ and $t \in (0, \infty)$, then x = y.
Notation: Φ is the class of all mappings $\xi : [0, 1] \to [0, 1]$ satisfying the following conditions:
(i) ξ is increasing on [0, 1] and
(ii) $\xi(t) > t, \forall\ t \in (0, 1)$ and $\xi(t) = t$ *iff* $t = 1$.
Gupta *et al*., [9] proved the following Theorem

Theorem 1.8 Let $(X, M, *)$ be a complete fuzzy metric space $\xi \in \Phi, k \in (0, 1)$ and $f: X \to X$ be a mapping satisfying
$$M(fx, fy, kt) \geq \xi\{\lambda(x, y, t)\}, \forall x, y \in X \tag{1.8.1}$$

where
$$\lambda(x, y, t) = \min\{ M(x, y, t), M(x, fx, t), \frac{M(y, fy, t)[1 + M(x, fx, t)}{[1 + M(x, y, t)} \} \tag{1.8.2}$$
Then f has a unique fixed point. ■
From this theorem, the following corollary is claimed an immediate consequence. Gupta [9]

Corollary 1.9 Let $(X, M, *)$ be a complete fuzzy metric space and $f: X \to X$ be a mapping satisfying
$$M(fx, fy, kt) \geq \lambda(x, y, t), \forall x, y \in X \text{ and } t > 0, k \in (0, 1). \tag{1.9.1}$$

where
$$\lambda(x, y, t) = \min\{ M(x, y, t), M(x, fx, t), \frac{M(y, fy, t)[1 + M(x, fx, t)}{[1 + M(x, y, t)} \}$$

Then f has a unique fixed point. ■
In Gupta [9] his claim that the corollary follows from Theorem 1.8 by taking $\xi(t) = t$ *for all* $t \in (0,1)$.
However if we take $\xi(t) = t$ *for all* $t \in (0,1)$ *then* $\xi \in \Phi$
In this paper we modify corollary 1.9 by replacing (1.9.1) by
$$M(fx, fy, kt) \geq \lambda(x, y, t) \text{ with equality iff } \lambda(x, y, t) = 1 \tag{1.9.1$^{\mathrm{I}}$}$$
In fact we prove a little more general result in Section 2

2. Main result
Now we state and prove our main result.

Theorem 2.1 Let $(X, M, *)$ be a complete fuzzy metric space $k \in (0, 1)$ and $f: X \to X$ be a mapping satisfying
$$M(fx, fy, kt) \geq \mu(x, y, t), \text{ with equality iff } \mu(x, y, t) = 1 \tag{2.1.1}$$

where
$$\mu(x, y, t) = \min\{ M(x, y, t), M(x, fx, t), M(x, fx, t)\frac{[1 + M(y, fy, t)]}{[1 + M(x, y, t)]}, M(y, fy, t), M(y, fy, t)\frac{[1 + M(x, fx, t)]}{[1 + M(x, y, t)]} \}$$
for all $x, y \in X$ *and* $t > 0$. Then f has a unique fixed point.

Proof: Let $x_0 \in X, x_1 = fx_0,\ x_n = fx_{n-1}, n = 1, 2, 3, \dots$
$$M(x_n, x_{n+1}, kt) = M(fx_{n-1}, fx_n, kt) \geq \mu(x_{n-1}, x_n, t)$$

where

$$\mu(x_{n-1},x_n,t) = \min\left\{\begin{matrix} M(x_{n-1},x_n,t), M(x_{n-1},fx_{n-1},t)\dfrac{[1+M(x_n,fx_n,t)]}{1+M(x_{n-1},x_n,t)}, M(x_{n-1},fx_{n-1},t), M(x_n,fx_n,t), \\ M(x_n,fx_n,t)\dfrac{[1+M(x_{n-1},fx_{n-1},t)}{1+M(x_{n-1},x_n,t)} \end{matrix}\right\}$$

$$= \min\left\{\begin{matrix} M(x_{n-1},x_n,t), M(x_{n-1},x_n,t)\dfrac{[1+M(x_n,x_{n+1},t)]}{1+M(x_{n-1},x_n,t)}, M(x_{n-1},x_n,t), M(x_n,x_{n+1},t), \\ M(x_n,x_{n+1},t)\dfrac{[1+M(x_{n-1},x_n,t)}{1+M(x_{n-1},x_n,t)} \end{matrix}\right\}$$

$$= \min\left\{M(x_{n-1},x_n,t), M(x_{n-1},x_n,t)\frac{[1+M(x_n,x_{n+1},t)]}{1+M(x_{n-1},x_n,t)}, M(x_n,x_{n+1},t)\right\}$$

$$= \min\{M(x_{n-1},x_n,t), M(x_n,x_{n+1},t)\}$$

Therefore
$M(x_n,x_{n+1},kt) \geq \min\{M(x_{n-1},x_n,t), M(x_n,x_{n+1},t)\}$
If $M(x_n,x_{n+1},t) < M(x_{n-1},x_n,t)$, $then\ \mu(x_n,x_{n+1},t) = M(x_n,x_{n+1},t) < M(x_{n-1},x_n,t) \leq 1$
Then $M(x_n,x_{n+1},k\,t) > \mu(x_{n-1},x_n,t) = M(x_n,x_{n+1},t)$

This implies that $M(x_n,x_{n+1},k\,t) > M(x_n,x_{n+1},t)$, a contradiction.
Hence $M(x_n,x_{n+1},t) \geq M(x_{n-1},x_n,t)$ for every t > 0.
Therefore $M(x_n,x_{n+1},t) \geq \mu(x_{n-1},x,t) = M(x_{n-1},x_n,t)$.

Suppose $\lim_{n\to\infty} M(x_n,x_{n+1},t) = \alpha\,.Then\ M(x_n,x_{n+1},t) \leq \alpha \leq 1\ for\ every\ n.$
Therefore $M(x_n,x_{n+1},t) \geq \mu(x_{n-1},x_n,\frac{t}{k}) = M\left(x_{n-1},x_n,{}^{t}/_{k}\right)\ for\ every\ n.$
By induction $M(x_n,x_{n+1},t) \geq M\left(x_0,x_1,{}^{t}/_{k^n}\right) \to 1\ as\ n \to \infty.$
Therefore $\alpha = 1\ so\ that\ \lim_{n\to\infty} M(x_n,x_{n+1},t) = 1.$
Now let p be any positive integer.
Then $M(x_n,x_{n+p},t) = M(x_n,x_{n+p},\frac{t}{p}+\frac{t}{p}+\cdots+\frac{t}{p}\ (p-times))$
$\geq M(x_n,x_{n+1},\frac{t}{p}) * M\left(x_n,x_{n+2},\frac{t}{p}\right) * \ldots * M(x_{n+p-1},x_{n+p},\frac{t}{p}) \to 1 * 1 * \ldots\ldots\ldots * 1 = 1$

Therefore $M(x_n,x_{n+p},t) \to 1\ as\ n \to \infty$ for each positive integer p.
Therefore $\{x_n\}$ is a Cauchy sequence.
Since $(X,M,*)$ is complete, there exists $x \in X\ such\ that\ x_n \to x\ as\ n \to \infty$
Therefore $\lim_{n\to\infty} M(x_n,x,t) = 1$

Claim: x is a fixed point of f.
$M(x_{n+1},fx,kt) = M(fx_n,fx,kt) \geq \mu(x_n,x,t)$ (2.1.2)

where

$$\mu(x_n,x,t) = \min\left\{\begin{matrix} M(x_n,x,t), M(x_n,fx_n,t)\dfrac{[1+M(x,fx,t)]}{1+M(x_n,x,t)}, M(x_n,fx_n,t), M(x,fx,t), \\ M(x,fx,t)\dfrac{[1+M(x_n,fx_n,t)}{1+M(x_n,x,t)} \end{matrix}\right\}$$

$$\to M(x,fx,t)\ as\ n \to \infty$$

From (2.1.2) we get $M(x,fx,kt) \geq M(x,fx,t)$. This is true for every t > 0.
Therefore $M(x,fx,t) = 1. Hence\ fx = x\ from\ (vi)$.
Therefore x is a fixed point of f.

Uniqueness: Let $x\ and\ y$ be two fixed points.
$M(x,y,kt) = M(fx,fy,kt) \geq \mu(x,y,t)\ with\ equlity\ iff\ \mu(x,y,t) = 1$

$$\mu(x,y,t) = min\left\{\begin{array}{c} M(x,y,t), M(x,fx,t)\dfrac{[1+M(y,fy,t)]}{1+M(x,y,t)}, M(x,fx,t), M(y,fy,t), \\ M(y,fy,t)\dfrac{[1+M(x,fx,t)}{1+M(x,y,t)} \end{array}\right\}$$

$= min\left\{M(x,y,t), \frac{2}{1+M(x,y,t)}, \frac{2}{1+M(x,y,t)}\right\}$
$= M(x,y,t)$

Therefore $M(x,y,kt) \geq M(x,y,t)\ with\ equlity\ iff\ \mu(x,y,t) = 1$
This is true for all t > 0. Hence $x = y$.

Corollary 2.2: Theorem 2.1 implies Corollary 1.9 of Gupta et. al., [9] with (1.9.1) replace by (1.9.1)[I]
The proof follows as $\lambda(x,y,t) \geq \mu(x,y,t)\ with\ equlity\ iff\ \lambda(x,y,t) = 1$
Hence result follows.

Corollary 2.3: Theorem 1.8.
This follow since $\xi\big(\lambda(x,y,t)\big) \geq \xi\big(\mu(x,y,t)\big) \geq \mu(x,y,t)\ with\ inequality\ iff\ \mu(x,y,t) = 1$

3. Application

Let φ be a non-negative L-Integrable function on [0, 1] such that $\int_a^b \varphi(s)ds > 0$ whenever $0 \leq a < b$. Let us define $\psi : [0,1] \rightarrow [0,1]$ by $\psi(t) = \int_0^t \varphi(s)ds$ for each t > 0. Then ψ is a strictly increasing continuous function.

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and $f: X \rightarrow X$ be a mapping satisfying

$$\int_0^{M(fx,fy,kt)} \varphi(s)ds \geq \int_0^{\mu(x,y,t)} \varphi(s)ds \qquad (3.1.1)$$

where

$$\mu(x,y,t) = min\left\{\begin{array}{c} M(x,y,t), M(x,fx,t), M(x,fx,t)\dfrac{[1+M(y,fy,t)]}{[1+M(x,y,t)]}, M(y,fy,t), \\ M(y,fy,t)\dfrac{[1+M(x,fx,t)]}{[1+M(x,y,t)]} \end{array}\right\}$$

for all $x, y \in X, t > 0$. Then f has a unique fixed point.

Proof: From (3.1.1) follows that $\int_{\mu(x,y,t)}^{M(fx,fy,kt)} \varphi(s)ds \geq 0$ for every t > 0.
Hence (2.1.1) holds. Therefore by theorem2.1 f has a unique fixed point.

References

1. Banach S. Sur les oprations dans les ensembles abstraits et leur application aux quations integrals [On operations in the abstract sets and their application to the integral equations. Fundamenta Mathematicae, 1992; 3:133-181.
2. El Naschie MS. A review of E-infinity theory and the mass spectrum of high energy particle physics. Chaos Solitons Fractals, 2004; 19:209-236.
3. George A, Veeramani P. On some results in fuzzy metric spaces. Fuzzy Sets and Systems, 1994; 64:395-399.
4. Grabiec M. Fixed point in fuzzy metric spaces. Fuzzy Sets and Systems, 1988; 27:385-389.
5. Gregori V, Morillas S, Sapena A. Examples of fuzzy metrics and applications. Fuzzy Sets and Systems, 2011; 170:95-111.
6. Gregori V, Sapena A. On fixed-point theorems in fuzzy metric spaces. Fuzzy Sets and Systems, 2002; 125:245-252.
7. Gupta V, Mani N. Existence and uniqueness of fixed point in fuzzy metric spaces and its applications. Proceedings of the Second International Conference on Soft Computing for Problem Solving: Advances in Intelligent Systems and Computing, 2014a; 236:217-224.
8. Gupta V, Mani N. Common fixed points by using E.A. property in fuzzy metric spaces. Proceedings of the Third International Conference on Soft Computing for Problem Solving: Advances in Intelligent Systems and Computing, 2014b; 259:45-54.
9. Gupta V, Saini RK, Navven Mani, Adesh kumar Tripathi. Cogent Mathematics 2015; 2:1053173
10. Kramosil I, Michalek J. Fuzzy metric and statistical metric spaces, Ky-bernatica, 1975; 11:326-334.
11. Saini RK, Gupta V, Singh SB. Fuzzy version of some fixed points theorems on expansopm type maps in fuzzy metric space. Thai Journal of Mathematocs, 2007; 5:245-252.
12. Saini RK, Kumar M, Gupta V, Singh SB. Common coincidence points of R-weakly commuting fuzzy maps. Thai Journal of Mathematics. 2008; 6:109-115.
13. Schwizer B, sklar A. Statistical metric spaces. Pacific Journal of Mathematics. 1960; 10:313-334.
14. Subrahmanyam PV. A common fixed point theorem in fuzzy metric space. Information Sciences, 1995; 83:109-112.
15. Vasuki R. A common fixed point theorem in fuzzy metric space. Fuzzy Sets and Systems, 1988; 97:395-397.

16. Vijayaraju P, Sajath ZMI. Some common fixed point theorems in fuzzy metric spaces. International Journal of Mathematical Analysis. 2009; 3:701-710.
17. Zadeh LA. Fuzzy sets. Information and Control, 1965; 8:338-353.