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Generalization of Fuzzy Semi Boundary in Fuzzy Bitopological Spaces

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Abstract
In this paper we genaralize the concept of fuzzy τ_i - semi closure, τ_i - semi interior and (τ_i, τ_j) - semi boundary namely $\mathfrak{C}$- τ_i - semi closure, $\mathfrak{C}$ -τ_i - semi interior and $\mathfrak{C}$ -(τ_i, τ_j) - semi boundary subsets in a fuzzy bitopological space where $\mathfrak{C} : [0,1] \to [0,1]$ is a complement function. Several examples are given to illustrate the concepts introduced in this paper.

Keywords: Fuzzy complement function $\mathfrak{C}$, $\mathfrak{C}$ - τ_i, – semi closure, $\mathfrak{C}$ - τ_i, - semi interior, fuzzy $\mathfrak{C}$-(τ_i, τ_j) - semi boundary subsets and fuzzy bitopological spaces.

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh [13] in the year 1965. The theory of fuzzy topological space was introduced and developed by C. L. Chang [6]. A. Kandil [10] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological space. The concept of complement function $\mathfrak{C}$: $[0,1] \to [0,1]$ was introduced by K. Bageerathi and P. Thangavelu in [4]. The concept of fuzzy boundary was introduced and studied byPu and Liu in [11]. The concept of fuzzy boundary have been studied in Athar and Ahmad [3, 1, 2, 7, 8, 13]. The concept of fuzzy $\mathfrak{C}$-(τ_i, τ_j) - boundary subset introduced and studied in [12]. In this paper the concept of fuzzy $\mathfrak{C}$- τ_i, -semi closure, fuzzy $\mathfrak{C}$- τ_i, -semi interior and fuzzy $\mathfrak{C}$-(τ_i, τ_j) -semi boundary subset introduced and several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

In this section we list some definitions and results that are needed. Any function $\mathfrak{C}$: $[0, 1] \to [0, 1]$ defined from the interval [0, 1] to itself is called a complement function. Throughout the paper $\mathfrak{C}$ denotes an arbitrary complement function and (X, τ_i, τ_j) is a fuzzy bitopological space in the sense of A. Kandil [10]. For the definitions and results that are not mentioned in this paper, the reader can refer K. Bageerathi [5].Throughout thispaper, for fuzzy set A of a fuzzy bitopological space (X, τ_i, τ_j), τ_i - intA and τ_j - $cl_{\mathfrak{C}}A$ means, respectively, the interior and closure of A with respect to fuzzy topologies τ_i andτ_j. Throughout this paper, $\mathfrak{C}$ - τ_i - semi open sets and $\mathfrak{C}$ - τ_j - semi closed means, respectively, the semi open sets and semi closed set with respect to fuzzy topologies τ_i andτ_j sets of X.

Definition 2.1[4]

If A is a fuzzy subset of X then the complement $\mathfrak{C}A$ of a fuzzy set A is a fuzzy subset with membership function defined by $\mu_{\mathfrak{C}A}(x) = \mathfrak{C}(\mu_A(x))$ for all $x \in X$.
A subset A of a fuzzy topological space is fuzzy closed if its standard complement λ', where $\lambda'(x) = 1-\lambda(x)$ is fuzzy open. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements available in the fuzzy literature.
The properties of fuzzy complement function $\mathfrak{C}$ and $\mathfrak{C}\lambda$ are given in George Klir and Bageerathi *et al* [9, 4]. The following lemma will be useful in sequel. Some of the complement functions are given below.

Examples 2.2 [9]

(i) The standard complement function: $\mathfrak{C}_1(x) = 1 - x$.

(ii) The Threshold type complement function for any $t \in [0, 1)$:

$$\mathfrak{C}_t(x) = \begin{cases} 1 & \text{for } 0 \le x \le t \\ 0 & \text{for } t < x \le 1 \end{cases}.$$

(iii) Sugeno class complement function for any $\lambda \in (1, \infty)$:

$$\mathfrak{C}_{S\lambda}(x) = \frac{1-x}{1+\lambda x}, \text{ for } x \in [0, 1].$$

(iv) Yagor class of complement function for $\omega \in (o, \infty)$:

$$\mathfrak{C}_{Y\omega}(x) = (1 - x^{\omega})^{1/\omega}, \text{ for } x \in [0, 1].$$

The next lemma can be easily established.

Lemma 2.3[9]

The complement functions $\mathfrak{C}_1$, $\mathfrak{C}_t$, $\mathfrak{C}_{S\lambda}$ and $\mathfrak{C}_{Y^\omega}$ satisfy the following conditions.

(i) Boundary condition: $\mathfrak{C}(0) = 1$ and $\mathfrak{C}(1) = 0$;

(ii) Monotonicity: for all $x, y \in [0, 1]$, $x \le y \Rightarrow \mathfrak{C}(x) \ge \mathfrak{C}(y)$;

(iii) $\mathfrak{C}$ is continuous and

(iv) Involutive: $\mathfrak{C}(\mathfrak{C}(x)) = x$ for all $x \in [0, 1]$.

Definition 2.4[1]

For a family $\{A_\alpha : \alpha \in \Delta\}$ of fuzzy sub sets of X, the union, $A = \cup\{A_\alpha : \alpha \in \Delta\}$ and the intersection, $B = \cap\{A_\alpha : \alpha \in \Delta\}$ are defined with membership functions respectively

$\mu_A(x) = \sup\{\mu_{A\alpha}(x) : \alpha \in \Delta\}$ and $\mu_B(x) = \inf\{\mu_{A\alpha}(x) : \alpha \in \Delta\}$, $x \in X$.

Lemma 2.5[4]

Let $\mathfrak{C}: [0, 1] \to [0, 1]$ be a complement function that satisfies the involutive and monotonicity properties. Then for any family $\{A_\alpha : \alpha \in \Delta\}$ of fuzzy subsets of X we have

(i) $\mathfrak{C}(\sup\{\mu_{A\alpha}(x) : \alpha \in \Delta\} = \inf\{\mathfrak{C}(\mu_{A\alpha}(x)) : \alpha \in \Delta\} = \inf\{(\mu_{\mathfrak{C}A\alpha}(x)) : \alpha \in \Delta\}$ and

(ii) $\mathfrak{C}(\inf\{\mu_{A\alpha}(x) : \alpha \in \Delta\} = \sup\{\mathfrak{C}(\mu_{A\alpha}(x)) : \alpha \in \Delta\} = \sup\{(\mu_{\mathfrak{C}A\alpha}(x)) : \alpha \in \Delta\}$.

Lemma 2.6[4]

Let $\mathfrak{C}: [0, 1] \to [0, 1]$ be a complement function that satisfies involutive and monotonicity properties. Then for any family $\{A_\alpha : \alpha \in \Delta\}$ of fuzzy subsets of X. we have

(i) $\mathfrak{C}(\cup\{A_\alpha : \alpha \in \Delta\}) = \cap\{\mathfrak{C}A_\alpha : \alpha \in \Delta\}$ and (ii) $\mathfrak{C}(\cap\{A_\alpha : \alpha \in \Delta\}) = \cup\{\mathfrak{C}A_\alpha : \alpha \in \Delta\}$.

3. Fuzzy $\mathfrak{C}$ - τ_i - semi interior and Fuzzy $\mathfrak{C}$ - τ_i - semi closure

In this section we define the notion of fuzzy $\mathfrak{C}$- τ_i - semi interior and fuzzy $\mathfrak{C}$- τ_i - semi closure operators and discussed some of their properties.

Definition 3.1

Let (X, τ_i, τ_j) be a fuzzy bitopological space and $\mathfrak{C}$ be a complement function. Then for a fuzzy subset λ of X, the fuzzy $\mathfrak{C}$ - τ_i - semi interior of λ (briefly τ_i - SInt $_{\mathfrak{C}}\lambda$), is the union of all fuzzy $\mathfrak{C}$ - τ_i - semi open sets of X contained in λ.

That is τ_i - SInt $_{\mathfrak{C}}(\lambda) = \vee\{\mu : \mu \le \lambda, \mu$ is fuzzy $\mathfrak{C}$ - τ_i - semi open$\}$.

Definition 3.2

Let (X, τ_i, τ_j) be a fuzzy bitopological space and $\mathfrak{C}$ be a complement function. Then for a fuzzy subset λ of X, the fuzzy $\mathfrak{C}$ - τ_i - semi closure of λ (briefly τ_i - S cl$_{\mathfrak{C}}$ λ), is the intersection of all fuzzy $\mathfrak{C}$ - τ_i - semi closed sets of X containing λ.

That is τ_i - S cl$_{\mathfrak{C}}(\lambda) = \wedge\{\mu : \mu \ge \lambda, \mu$ is fuzzy $\mathfrak{C}$ - τ_i - semi closed$\}$.

The concepts of fuzzy $\mathfrak{C}$ - τ_i - semi closure and fuzzy τ_i - semi closure are identical if $\mathfrak{C}$ is the standard complement function.

Theorem 3.3

Let (X, τ_i, τ_j) be a fuzzy bitopological space and $\mathfrak{C}$ be any complement function. Then for a fuzzy subset λ and μ of a fuzzy bitopological space X, we have (i) τ_i - SInt$_{\mathfrak{C}}(\lambda) \le \lambda$. (ii) λ is fuzzy $\mathfrak{C}$ - τ_i - semi open $\Leftrightarrow \tau_i$ - SInt$_{\mathfrak{C}}(\lambda) = \lambda$. (iii) τ_i - SInt$_{\mathfrak{C}}(\tau_i$ - SInt$_{\mathfrak{C}}(\lambda)) = \tau_i$ - SInt$_{\mathfrak{C}}(\lambda)$. (iv) If $\lambda \le \mu$ then τ_i - SInt$_{\mathfrak{C}}(\lambda) \le \tau_i$ - SInt$_{\mathfrak{C}}(\mu)$.

Theorem 3.4

Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for a fuzzy subset λ of a fuzzy bitopological space X, we have (i) $\mathfrak{C}(\tau_i$ - SInt $_{\mathfrak{C}}(\lambda)) = \tau_i$ - S cl$_{\mathfrak{C}}(\mathfrak{C}\lambda)$. (ii) $\mathfrak{C}(\tau_i$ – S cl$_{\mathfrak{C}}(\lambda)) = \tau_i$ - SInt $_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

Theorem 3.5

Let (X, τ_i, τ_j) be a fuzzy bitopological space and $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for a fuzzy subset λ and μ of a fuzzy bitopological space X, we have (i) $\lambda \le \tau_i$ - S cl$_{\mathfrak{C}}(\lambda)$. (ii) λ is fuzzy $\mathfrak{C}$ - τ_i - semi closed $\Leftrightarrow \tau_i$ - S cl$_{\mathfrak{C}}(\lambda) = \lambda$. (iii) τ_i - S cl $_{\mathfrak{C}}(\tau_i$ - S cl$_{\mathfrak{C}}(\lambda)) = \tau_i$ - S cl$_{\mathfrak{C}}(\lambda)$. (iv) If $\lambda \le \mu$ then τ_i - S cl$_{\mathfrak{C}}(\lambda) \le \tau_i$ - S cl$_{\mathfrak{C}}(\mu)$.

4. Fuzzy $\mathfrak{C}$ - (τ_i, τ_j) - semi boundary.

In this section, the concept of fuzzy $\mathfrak{C}$ - (τ_i, τ_j) – semi boundary in fuzzy bitopological space is introduced and its properties are discussed.

Definition 4.1

Let λ be a fuzzy subset of a fuzzy bitopological space (X, τ_i, τ_j) and let $\mathfrak{C}$ be a complement function. Then the fuzzy $\mathfrak{C}$ - (τ_i, τ_j) - semi boundary of λ is defined as (τ_i, τ_j) -S Bd $_{\mathfrak{C}}(\lambda) = \tau_i$ - S cl$_{\mathfrak{C}}(\tau_j$ - S cl$_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S cl$_{\mathfrak{C}}(\tau_j$ - S cl$_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since arbitrary intersection of fuzzy $\mathfrak{C}$ - τ_i - semi closed sets is fuzzy $\mathfrak{C}$ - τ_i - semi closed, (τ_i, τ_j) -S Bd $_{\mathfrak{C}}\lambda$ is fuzzy $\mathfrak{C}$ - τ_i - semi closed.

Example 4.2

Let $X = \{a, b, c\}$, $\tau_1 = \{0, 1, \{c_{.3}\}, \{a_{.8}, b_{.7}\}, \{a_{.8}, b_{.7}, c_{.3}\}, 1\}$ and $\tau_2 = \{0, 1, \{c_{.9}\}, \{a_{.8}, b_{.9}\}, \{a_{.8}, b_{.9}, c_{.9}\}, \{a_{.9}, b_{.9}, c_{.9}\}\}$. Let $\mathfrak{C}(x) = \frac{1}{1+3x}$, $0 \le x \le 1$, be a complement function. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{1, \{a_{.25}, b_{.25}, c_{.25}\}, \{a_1, b_1, c_{.526}\}, \{a_{.294}, b_{.3225}, c_1\}, \{a_{.294}, b_{.3225}, c_{.526}\}\}$ and $\mathfrak{C}(\tau_2) = \{1, \{a_{.25}, b_{.25}, c_{.25}\}, \{a_1, b_1, c_{.27}\}, \{a_{.29}, b_{.27}, c_1\}, \{a_{.29}, b_{.27}, c_{.27}\}, \{a_{.27}, b_{.27}, c_{.27}\}\}$. Let $\lambda = \{a_{.2}, b_{.2}, c_{.8}\}$. Then τ_2 - S cl$_{\mathfrak{C}}(\lambda) = \{a_{.24}, b_{.25}, c_{.9}\}$, τ_1 - S cl$_{\mathfrak{C}}(\tau_2$ - S cl$_{\mathfrak{C}}(\lambda)) = \{a_{.25}, b_{.25}, c_{.9}\}$. Also $\mathfrak{C}\lambda = \{a_{.625}, b_{.625}, c_{.27}\}$. Then τ_2 - S cl$_{\mathfrak{C}}(\mathfrak{C}\lambda) =$

$\{a_{.7}, b_{.7}, c_{.27}\}$ and τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}, c_{.4}\}$. Then it can be evaluated that (τ_1, τ_2) - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.25}, b_{.25}, c_{.4}\}$.

Proposition 4.3

Let (X, τ_i, τ_j) be a fuzzy bitopological space and let $\mathfrak{C}$ be a complement function that satisfies the involutive property. Then for any fuzzy subset λ of (X, τ_i, τ_j), (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

Proof.

By using Definition 4.1,(τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}\lambda) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the involutive property, $\mathfrak{C}(\mathfrak{C}\lambda) = \lambda$, that implies (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}\mathfrak{C}(\mathfrak{C}\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$. Therefore (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

The following example shows that the involutive property cannot be dropped from the hypothesis of Proposition 4.3.

Example 4.4

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_{.1}, b_{.2}, c_{.4}\}, \{a_{.2}, b_{.1}, c_{.7}\}, \{a_{.1}, b_{.1}, c_{.4}\}, \{a_{.2}, b_{.2}, c_{.7}\}, \{a_{.7}, b_{.7}, c_{.7}\}, 1\}$ and $\tau_2 = \{0, \{a_{.1}, b_{.1}, c_{.2}\}, \{a_{.3}, b_{.3}, c_{.3}\}, \{a_{.5}, b_{.5}, c_{.5}\}, 1\}$. Let $\mathfrak{C}(x) = \sqrt{x}$, $0 \le x \le 1$ be a complement function. We see that this complement function does not satisfy the involutive and monotonic properties. The family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{0, \{a_{.3}, b_{.4}, c_{.6}\}, \{a_{.4}, b_{.3}, c_{.8}\}, \{a_{.3}, b_{.3}, c_{.6}\}, \{a_{.4}, b_{.4}, c_{.8}\}, \{a_{.8}, b_{.8}, c_{.8}\}, 1\}$ and $\mathfrak{C}(\tau_2) = \{0, \{a_{.3}, b_{.3}, c_{.4}\}, \{a_{.5}, b_{.5}, c_{.5}\}, \{a_{.7}, b_{.7}, c_{.7}\}, 1\}$. Let $\lambda = \{a_{.1}, b_0, c_{.3}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.2}, b_0, c_{.4}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.2}, b_{.2}, c_{.5}\}$. Also $\mathfrak{C}\lambda = \{a_{.3}, b_0, c_{.5}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda) = \{a_{.3}, b_{.4}, c_{.5}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.3}, b_{.4}, c_{.6}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.2}, b_{.2}, c_{.5}\}$. Now $\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.55}, b_0, c_{.71}\}$, τ_2 - S $cl_{\mathfrak{C}}\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.7}, b_{.7}, c_{.71}\}$ and τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.7}, b_{.7}, c_{.8}\}$. Then $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda) = \{a_{.3}, b_{.4}, c_{.6}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) \ne (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

Proposition 4.5

Let (X, τ_i, τ_j) be a fuzzy bitopological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and Involutive properties. If λ is fuzzy $\mathfrak{C}$ - τ_i – semi closed, i = 1, 2 then (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) \le \lambda$.

Proof.

Let λ be a fuzzy $\mathfrak{C}$ - τ_i - semi closed, i = 1, 2. By using Definition 4.1,(τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5, we have τ_i - S $cl_{\mathfrak{C}}\lambda = \lambda$. Hence (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) \le \tau_i$ - S $cl_{\mathfrak{C}}(\lambda) = \lambda$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive conditions, then the conclusion of Proposition 4.5 is false.

Example 4.6

Let $X = \{a, b\}$, $\tau_1 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.2}\}, \{a_{.6}, b_{.2}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$ and $\tau_2 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{2x}{1+x}$, $0 \le x \le 1$, be a complement function. We see that this complement function does not satisfy the involutive and monotonic properties. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.333}\}, \{a_{.8}, b_{.333}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$ and $\mathfrak{C}(\tau_2) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.461}\}, \{a_{.8}, b_{.461}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$. Let $\lambda = \{a_{.7}, b_{.7}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.7}, b_{.7}\}$ and τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.7}, b_{.71}\}$. Also $\mathfrak{C}\lambda = \{a_{.8}, b_{.8}\}$. Then it can be evaluated that τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.7}, b_{.71}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) \nleq \lambda$.Also λ is fuzzy $\mathfrak{C}$ - τ_i - semi closed, i=1,2.

Proposition 4.7

Let (X, τ_i, τ_j) be a fuzzy bitopological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. If λ is fuzzy $\mathfrak{C}$ - τ_i – semi open, i = 1, 2 then (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda \le \mathfrak{C}\lambda$.

Proof.

Let λ be a fuzzy $\mathfrak{C}$ - τ_i – semi open, i = 1, 2. Since $\mathfrak{C}$ satisfies the involutive property, this implies that $\mathfrak{C}(\mathfrak{C}\lambda)$ is fuzzy $\mathfrak{C}$ - τ_i -semi open. Therefore $\mathfrak{C}\lambda$ is a fuzzy $\mathfrak{C}$ - τ_i – semi closed. Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Proposition 4.5, (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\mathfrak{C}\lambda) \le \mathfrak{C}\lambda$. Also by using Proposition 4.3, we get (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda \le \mathfrak{C}\lambda$. Hence the proof.

Example 4.8

Let $X = \{a, b\}$, $\tau_1 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.2}\}, \{a_{.6}, b_{.2}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$ and $\tau_2 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{2x}{1+x}$, $0 \le x \le 1$, be a complement function. We see that this complement function does not satisfy the involutive and monotonic properties. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.333}\}, \{a_{.8}, b_{.333}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$ and $\mathfrak{C}(\tau_2) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.461}\}, \{a_{.8}, b_{.461}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$. Let $\lambda = \{a_{.6}, b_{.6}\}$. Then λ is $\mathfrak{C}$ - τ_i - semi open. It can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.7}, b_{.6}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.7}, b_{.76}\}$. Also $\mathfrak{C}\lambda = \{a_{.75}, b_{.75}\}$. Now τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda) = \{a_{.8}, b_{.8}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.7}, b_{.76}\} \nleq \{a_{.75}, b_{.75}\} = \mathfrak{C}\lambda$.

Proposition 4.9

Let (X, τ_i, τ_j) be a fuzzy bitopological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. If $\lambda \le \mu$ and μ is fuzzy $\mathfrak{C}$ - τ_i – semi closed then (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda \le \mu$.

Proof.

Let $\lambda \le \mu$ and μ be a fuzzy $\mathfrak{C}$ - τ_i -semi closed. Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5, we have $\lambda \le \mu$ implies τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \le \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))$. By using Definition 4.1, (τ_i, τ_j) - S $Bd_{\mathfrak{C}}\lambda = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - $SCl_{\mathfrak{C}}\lambda) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \le \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))$ we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) \le \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) \le (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu)$. Again by using Theorem 3.5, we have τ_i - S $cl_{\mathfrak{C}}(\mu) = \mu$. This implies that (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) \le \mu$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.9 is false.

Example 4.10

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_0, b_0, c_{.4}\}, \{a_{.7}, b_0, c_0\}, \{a_{.7}, b_0, c_{.4}\}, 1\}$ and $\tau_2 = \{0, \{a_0, b_0, c_{.5}\}, \{a_{.6}, b_0, c_0\}, \{a_{.6}, b_0, c_{.5}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{1-x}{(1+x)^2}$, be a complement function that does not satisfy the involutive condition. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.306}\}, \{a_{.104}, b_1, c_1\}, \{a_{.104}, b_1, c_{.306}\}, 0\}$ and $\mathfrak{C}(\tau_2) = \{\{a_1, b_1, c_{.2}\}, \{a_{.16}, b_1, c_1\}, \{a_{.16}, b_1, c_{.2}\}, 0\}$. Let $\mu = \{a_{.1}, b_{.8}, c_{.8}\}$ be $\mathfrak{C}$- τ_i - semi closed,i=1,2. Let $\lambda = \{a_{.1}, b_{.7}, c_{.7}\}$. Then it can be evaluated that τ_2 - S cl$_\mathfrak{C}(\lambda) = \{a_{.1}, b_{.9}, c_{.7}\}$ and τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\lambda)) = \{a_{.1}, b_{.9}, c_{.9}\}$. Also $\mathfrak{C}\lambda = \{a_{.7}, b_{.1}, c_{.1}\}$. Then it can be evaluated that τ_2 - S cl$_\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.7}, b_{.1}, c_{.1}\}$, τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.2}, c_{.2}\}$. This implies that $(\tau_1 - \tau_2)$ - S Bd$_\mathfrak{C}(\lambda) = \{a_{.1}, b_{.9}, c_{.2}\} \not\leq \{a_{.1}, b_{.8}, c_{.8}\}$.

Proposition 4.11

Let (X, τ_i, τ_j) be a fuzzy bitopological space and let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. If $\lambda \leq \mu$ and μ is fuzzy $\mathfrak{C}$ - τ_i -semi open then (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda \leq \mathfrak{C}\mu$.

Proof.

Let $\lambda \leq \mu$ and μ be a fuzzy $\mathfrak{C}$ - τ_i -semi open. Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Proposition 3.5, we have $\mathfrak{C}\mu \leq \mathfrak{C}\lambda$ that implies τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\mu$)) $\leq \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)). By using Definition 4.1, (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda = \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)). Taking complement on both sides, we get $\mathfrak{C}$ ((τ_i, τ_j) - S Bd$_\mathfrak{C}$ (λ)) = $\mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$))). Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Lemma 2.6, we have $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) = $\mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\vee\ \mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$))). Since τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\mu$)) $\leq \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)), $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) $\geq \mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\vee\ \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\mu$))). By using Lemma 3.4, $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) $\geq$ (τ_i - SInt (τ_j - SInt($\mathfrak{C}\lambda$)) $\vee\ \tau_i$ - SInt (τ_j - SInt (μ))). Since μ is fuzzy $\mathfrak{C}$ - τ_i -semi open, $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) $\geq \mu$. Since $\mathfrak{C}$ satisfies the monotonic properties, (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda \leq \mathfrak{C}\mu$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.11 is false.

Example 4.12

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_0, b_0, c_{.4}\}, \{a_{.7}, b_0, c_0\}, \{a_{.7}, b_0, c_{.4}\}, 1\}$ and $\tau_2 = \{0, \{a_0, b_0, c_{.5}\}, \{a_{.6}, b_0, c_0\}, \{a_{.6}, b_0, c_{.5}\}, 1\}$.Let $\mathfrak{C}(x) = \frac{1-x}{(1+x)^2}$ be a complement function that does not satisfy the involutive property. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are $\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.306}\}, \{a_{.104}, b_1, c_1\}, \{a_{.104}, b_1, c_{.306}\}, 0\}$ and $\mathfrak{C}(\tau_2) = \{\{a_1, b_1, c_{.2}\}, \{a_{.16}, b_1, c_1\}, \{a_{.16}, b_1, c_{.2}\}, 0\}$. Let $\mu = \{a_{.7}, b_{.1}, c_{.6}\}$. Then μ is fuzzy $\mathfrak{C}$- τ_i - semi open. Let $\lambda = \{a_{.7}, b_{.1}, c_{.1}\}$. Then it can be evaluated that τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\lambda)) = \{a_{.8}, b_{.9}, c_{.2}\}$. Now$\mathfrak{C}\lambda = \{a_{.1}, b_{.7}, c_{.7}\}$. Then τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.1}, b_{.9}, c_{.9}\}$. This implies that $(\tau_1 - \tau_2)$ - S Bd$_\mathfrak{C}(\lambda) = \{a_{.1}, b_{.9}, c_{.2}\}$. Therefore $(\tau_1 - \tau_2)$ - S Bd$_\mathfrak{C}(\lambda) \not\leq \mathfrak{C}\mu = \{a_{.1}, b_{.7}, c_{.2}\}$.

Proposition 4.13

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any fuzzy subset λ of X, we have $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) = (τ_i - SInt (τ_j - SInt(λ)) $\vee\ \tau_i$ - SInt (τ_j - SInt ($\mathfrak{C}\lambda$))).

Proof.

By using Definition 4.1, (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda = \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \tau_i$ - SCl $_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)). Taking complement on both sides, we get $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) = $\mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)))). Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5, we have $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) = $\mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$)) $\vee\ \mathfrak{C}$ (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$))). Also by using Theorem 3.4, $\mathfrak{C}$ ((τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$) = (τ_i - SInt (τ_j - SInt($\mathfrak{C}\lambda$)) $\vee\ \tau_i$ - SInt (τ_j - SInt (λ))).

The following example shows that if the monotonic and involutive conditions of the complement function $\mathfrak{C}$ can be dropped, then the conclusion of Proposition 4.13 is false.

Example 4.14

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_{.1}, b_{.2}, c_{.3}\}, \{a_{.5}, b_{.7}, c_{.8}\}, 1\}$ and $\tau_2 = \{0, \{a_{.2}, b_{.3}, c_{.1}\}, \{a_{.1}, b_{.5}, c_{.6}\}, \{a_{.1}, b_{.3}, c_{.1}\}, \{a_{.2}, b_{.5}, c_{.6}\}, 1\}$. Let $\mathfrak{C}(x) = \sqrt{x}$ be a complement function that does not satisfy the monotonic and involutive properties. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are given by$\mathfrak{C}(\tau_1) = \{0, \{a_{.3}, b_{.4}, c_{.5}\}, \{a_{.7}, b_{.8}, c_{.9}\}, 1\}$ and $\mathfrak{C}(\tau_2) = \{0, \{a_{.4}, b_{.5}, c_{.3}\}, \{a_{.3}, b_{.7}, c_{.8}\}, \{a_{.3}, b_{.5}, c_{.3}\}, \{a_{.4}, b_{.7}, c_{.8}\}, 1\}$. Let $\lambda = \{a_{.2}, b_{.3}, c_{.2}\}$. Then it can be evaluated that τ_2 - S cl$_\mathfrak{C}(\lambda) = \{a_{.2}, b_{.3}, c_{.2}\}$, τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\lambda)) = \{a_{.3}, b_{.3}, c_{.3}\}$. Now$\mathfrak{C}\lambda = \{a_{.4}, b_{.54}, c_{.4}\}$. Then τ_2 - S cl$_\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.4}, b_{.6}, c_{.7}\}$ and τ_1 - S cl$_\mathfrak{C}(\tau_2$ - S cl$_\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.4}, b_{.7}, c_{.7}\}$. This implies that $(\tau_1 - \tau_2)$ - S Bd$_\mathfrak{C}(\lambda) = \{a_{.3}, b_{.3}, c_{.3}\}$. $\mathfrak{C}$ (($\tau_1 - \tau_2$) - S Bd$_\mathfrak{C}(\lambda)) = \{a_{.5}, b_{.5}, c_{.5}\}$. Then it can be evaluated that τ_2 - S Int$_\mathfrak{C}(\lambda) = \{a_{.2}, b_{.3}, c_{.15}\}$, τ_1 - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\lambda)) = \{a_{.2}, b_{.3}, c_{.3}\}$. Now$\mathfrak{C}\lambda = \{a_{.4}, b_{.54}, c_{.4}\}$. Then τ_2 - S Int$_\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.2}, b_{.3}, c_{.2}\}$, τ_1 - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.2}, b_{.3}, c_{.2}\}$. This implies that τ_1 - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\lambda)) \vee \tau_1$ - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.2}, b_{.3}, c_{.3}\}$. Also $\mathfrak{C}((\tau_1 - \tau_2)$ - S Bd$_\mathfrak{C}(\lambda)) = \{a_{.5}, b_{.5}, c_{.5}\} \neq \{a_{.2}, b_{.3}, c_{.3}\} = \tau_1$ - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\lambda)) \vee \tau_1$ - S Int$_\mathfrak{C}(\tau_2$ - S Int$_\mathfrak{C}(\mathfrak{C}\lambda))$.

Proposition 4.15

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any fuzzy subset λ of X, we have (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda$ = (τ_i - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ (λ)) $\wedge\ \mathfrak{C}$ (τ_i – SInt (τ_j - SInt (λ))).

Proof.

By using Definition 4.1, we have (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda = \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}$ ($\mathfrak{C}\lambda$)). Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem3.4, we have (τ_i, τ_j) - S Bd $_\mathfrak{C}\lambda = \tau_i$ - S cl$_\mathfrak{C}$ (τ_j - S cl$_\mathfrak{C}\lambda$) $\wedge\ \mathfrak{C}$ (τ_i - SInt (τ_j - SInt(λ))).

The next example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.15 is false.

Example 4.16

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_0, b_0, c_{.4}\}, \{a_{.7}, b_0, c_0\}, \{a_{.7}, b_0, c_{.4}\}, 1\}$ and $\tau_2 = \{0, \{a_0, b_0, c_{.5}\}, \{a_{.6}, b_0, c_0\}, \{a_{.6}, b_0, c_{.5}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{1-x}{(1+x)^2}$ be a complement function that does not satisfy the involutive properties. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are given by$\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.306}\},$

$\{a_{.104}, b_1, c_1\}, \{a_{.104}, b_1, c_{.306}\}, 0\}$ and $\mathfrak{C}(\tau_2) = \{1, \{a_1, b_1, c_{.2}\}, \{a_{.16}, b_1, c_1\}, \{a_{.16}, b_1, c_{.2}\}, 0\}$. Let $\lambda = \{a_{.1}, b_{.1}, c_{.5}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.2}, c_{.5}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.1}, b_{.3}, c_{.6}\}$. Now $\mathfrak{C}\,\lambda = \{a_{.7}, b_{.7}, c_{.22}\}$. Then τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda) = \{a_{.7}, b_{.8}, c_{.7}\}$ and τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)) = \{a_{.8}, b_{.9}, c_{.8}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.3}, c_{.6}\}$. Then it can be evaluated that τ_2 - S $Int_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_0, c_{.5}\}$, τ_1 - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda)) = \{a_0, b_0, c_{.5}\}$. Also $\mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))) = \{a_1, b_1, c_{.22}\}$. This implies that τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))) = \{a_{.1}, b_{.3}, c_{.22}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.3}, c_{.6}\} \neq \{a_{.1}, b_{.3}, c_{.22}\} = \tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda)))$.

Proposition 4.17

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any fuzzy subset λ of X, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}\,\lambda$.

Proof.

Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Proposition 4.15, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda))) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt $(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))))$. Since τ_i - SInt $(\tau_j$ - SInt$(\lambda))$ is fuzzy $\mathfrak{C}$ - τ_i – semi open, (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda))) = \tau_i$ - $SCl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))$. Since τ_i - SInt $(\tau_j$ - SInt$(\lambda)) \leq \lambda$, by using Theorm 3.5, we have τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))) \leq \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))$. Thus (τ_i, τ_j)- S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda))) \leq \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda)))$. Since $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.4, we have (τ_i, τ_j) -S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda))) \leq \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda))$. By using Definition 4.1, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - SInt $(\tau_j$ - SInt$(\lambda))) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

The next example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.17 is false.

Example 4.18

In Example 3.16, $\lambda = \{a_{.1}, b_{.1}, c_{.5}\}$ implies that τ_1 - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda)) = \{a_0, b_0, c_{.5}\}$. Then τ_2 - S $cl_{\mathfrak{C}}\{\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))\} = \{a_{.1}, b_{.1}, c_{.8}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}\{\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))\}) = \{a_{.1}, b_{.8}, c_{.9}\}$. Now $\mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))) = \{a_1, b_1, c_{.22}\}$ then τ_2 - S $cl_{\mathfrak{C}}\{\mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda)))\} = \{a_1, b_1, c_{.23}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}\{\mathfrak{C}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda)))\}) = \{a_1, b_1, c_{.5}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))) = \{a_{.1}, b_{.8}, c_{.5}\} \nleq (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.3}, c_{.6}\}$.

Proposition 4.19

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then (τ_i, τ_j) -S $Bd_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

Proof.

Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Proposition 4.15, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)))) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt $(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)))))$. By using Theorem[3.4], we have τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)))) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))$ that implies (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt$(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)))))$. Since $\lambda \leq \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))$, that implies τ_i - SInt $(\tau_j$ - SInt $(\lambda)) \leq \tau_i$ - SInt $(\tau_j$ - SInt$(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))))$. Therefore (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) \leq \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_i$ - SInt $(\tau_j$ - SInt $(\lambda)))$. By using Theorem 3.4 and Definition 4.1, we get (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.19 is false.

Example 4.20

Let $X = \{a, b, c\}$, $\tau_1 = \{0, 1, \{c_{.3}\}, \{a_{.8}, b_{.7}\}, \{a_{.8}, b_{.7}, c_{.3}\}, 1\}$ and $\tau_2 = \{0, 1, \{c_{.9}\}, \{a_{.8}, b_{.9}\}, \{a_{.8}, b_{.9}, c_{.9}\}, \{a_{.9}, b_{.9}, c_{.9}\}\}$. Let $\mathfrak{C}(x) = \frac{1}{1+3x}$ be a complement function that does not satisfy the involutive property. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are given by $\mathfrak{C}(\tau_1) = \{1, \{a_{.25}, b_{.25}, c_{.25}\}, \{a_1, b_1, c_{.526}\}, \{a_{.294}, b_{.3225}, c_1\}, \{a_{.294}, b_{.3225}, c_{.526}\}\}$ and $\mathfrak{C}(\tau_2) = \{1, \{a_{.25}, b_{.25}, c_{.25}\}, \{a_1, b_1, c_{.27}\}, \{a_{.29}, b_{.27}, c_1\}, \{a_{.29}, b_{.27}, c_{.27}\}, \{a_{.27}, b_{.27}, c_{.27}\}\}$. Let $\lambda = \{a_{.2}, b_{.2}, c_{.8}\}$. Then it can be evaluated that τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.25}, b_{.25}, c_{.9}\}$. Now $\mathfrak{C}\,\lambda = \{a_{.625}, b_{.625}, c_{.27}\}$. Then τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)) = \{a_{.8}, b_{.8}, c_{.4}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.25}, b_{.25}, c_{.4}\}$. Let $\mu = \tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.25}, b_{.25}, c_{.9}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\mu) = \{a_{.25}, b_{.26}, c_{.9}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mu)) = \{a_{.25}, b_{.26}, c_{.91}\}$. Now $\mathfrak{C}\,\mu = \mathfrak{C}(\tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda))) = \{a_{.6}, b_{.6}, c_{.27}\}$. Then it can be calculated that τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu) = \{a_{.7}, b_{.6}, c_{.27}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu)) = \{a_{.7}, b_{.7}, c_{.4}\}$. This implies that τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)))) \wedge \tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,[\tau_1$ - S $Int_{\mathfrak{C}}(\tau_2$ - S $Int_{\mathfrak{C}}(\lambda))])) = \{a_{.25}, b_{.26}, c_{.91}\} \wedge \{a_{.7}, b_{.7}, c_{.4}\} = \{a_{.25}, b_{.26}, c_{.4}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda))) = \{a_{.25}, b_{.26}, c_{.4}\}$. But $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.25}, b_{.25}, c_{.4}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\tau_1$ - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda))) \nleq (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

Proposition 4.21

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Let $\mathfrak{C}$ be a complement function that satisfies the monotonic and involutive properties. Then (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda) \vee (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu)$.

Proof.

By using Definition 4.1, (τ_i, τ_j) -S $Bd_{\mathfrak{C}}(\lambda \vee \mu) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda \vee \mu)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu)))$. Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) = (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \vee \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu)))$. By using Lemma 2.6 and Theorem 3.5, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) \leq (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) \vee (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))) \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda))) \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu)))$. That implies (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) \leq (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda))) \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda))) \vee (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))) \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu)))$. By using Definition 4.1, (τ_i, τ_j) -S $Bd_{\mathfrak{C}}(\lambda \vee \mu) \leq ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) \vee ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu))$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.21 is false.

Example 4.22

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_0, b_0, c_{.4}\}, \{a_{.7}, b_0, c_0\}, \{a_{.7}, b_0, c_{.4}\}, 1\}$ and $\tau_2 = \{0, \{a_0, b_0, c_{.5}\}, \{a_{.6}, b_0, c_0\}, \{a_{.6}, b_0, c_{.5}\}, 1\}$. Let $\mathfrak{C}(x) = \frac{1-x}{(1+x)^2}$ be a complement function that does not satisfy the involutive property. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are given by $\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.306}\}, \{a_{.104}, b_1, c_1\}, \{a_{.104}, b_1, c_{.306}\}, 0\}$ and $\mathfrak{C}(\tau_2) = \{1, \{a_1, b_1, c_{.2}\}, \{a_{.16}, b_1, c_1\}, \{a_{.16}, b_1, c_{.2}\}, 0\}$. Let $\lambda = \{a_{.1}, b_0, c_0\}$ and $\mu = \{a_0, b_{.8}, c_{.8}\}$. Then $\lambda \vee \mu = \{a_{.1}, b_{.8}, c_{.8}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda \vee \mu) = \{a_{.1}, b_{.9}, c_{.8}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda \vee \mu)) = \{a_{.1}, b_{.9}, c_{.9}\}$. Now $\mathfrak{C}(\lambda \vee \mu) = \{a_{.7}, b_{.1}, c_{.1}\}$. Then τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu)) = \{a_{.7}, b_{.1}, c_{.1}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}(\lambda \vee \mu))) = \{a_{.8}, b_{.9}, c_{.2}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) = \{a_{.1}, b_{.9}, c_{.9}\} \wedge \{a_{.8}, b_{.9}, c_{.2}\} = \{a_{.1}, b_{.9}, c_{.2}\}$. Also τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.1}, c_0\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.1}, b_{.1}, c_{.2}\}$. Now $\mathfrak{C}\,\lambda = \{a_{.7}, b_1, c_1\}$. Then τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda) = \{a_{.8}, b_{.8}, c_{.8}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)) = \{a_{.9}, b_{.9}, c_{.9}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.1}, b_{.1}, c_{.2}\}$. Also $\mu = \{a_0, b_{.8}, c_{.8}\}$. Then τ_2 - S $cl_{\mathfrak{C}}(\mu) = \{a_0, b_{.9}, c_{.9}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mu)) = \{a_{.1}, b_{.9}, c_{.9}\}$. Also $\mathfrak{C}\,\mu = \{a_1, b_{.03}, c_{.03}\}$. Then τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu) = \{a_1, b_{.1}, c_{.1}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu)) = \{a_1, b_{.2}, c_{.1}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\mu) = \{a_{.1}, b_{.2}, c_{.1}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) \vee (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\mu) = \{a_{.1}, b_{.2}, c_{.2}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda \vee \mu) \nleq (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) \vee (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\mu)$.

Theorem 4.23

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Suppose the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties. Then for any fuzzy subset λ and μ of X, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \wedge \mu) \leq ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))) \vee ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)))$.

Proof.

By using Definition 4.1, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda \wedge \mu)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}(\lambda \wedge \mu)))$. Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5 and Lemma 2.6, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \wedge \mu) \leq (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu))) \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)) \vee \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu))) = [\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda))] \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu) \vee [\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mu)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu))] \wedge (\tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)$. By using Definition 4.1, (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda \wedge \mu) \leq ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) \wedge ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu)) \vee [((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) \wedge ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\mu))]$.

Theorem 4.24

Let (X, τ_i, τ_j) be a fuzzy bitopological space. Suppose the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties. Then for any fuzzy subset λ of X, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) \leq ((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

Proof.

By using Definition 4.1, we have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda))$. We have (τ_i, τ_j) - S $Bd_{\mathfrak{C}}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) = \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda))) \wedge \tau_i$ - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\mathfrak{C}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda))))$. Since the complement function $\mathfrak{C}$ satisfies the monotonic and involutive properties, by using Theorem 3.5, we have τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}(\lambda)) = \lambda$ where λ is fuzzy $\mathfrak{C}$ - τ_i – semi closed. Here (τ_i, τ_j) - S $Bd_{\mathfrak{C}}(\lambda)$ is fuzzy $\mathfrak{C}$ - τ_i -semi closed. We get τ_i - S $cl_{\mathfrak{C}}(\tau_j$ - S $cl_{\mathfrak{C}}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda))) = (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$. This implies that (τ_i, τ_j) - S $Bd_{\mathfrak{C}}((\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)) \leq (\tau_i, \tau_j)$ - S $Bd_{\mathfrak{C}}(\lambda)$. Hence the proof.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Theorem 4.24 is false.

Example 4.25

Let $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_{.1}, b_{.2}, c_{.4}\}, \{a_{.2}, b_{.1}, c_{.7}\}, \{a_{.1}, b_{.1}, c_{.4}\}, \{a_{.2}, b_{.2}, c_{.7}\}, \{a_{.7}, b_{.7}, c_{.7}\}, 1\}$ and $\tau_2 = \{0, \{a_{.1}, b_{.1}, c_{.2}\}, \{a_{.3}, b_{.3}, c_{.3}\}, \{a_{.5}, b_{.5}, c_{.5}\}, 1\}$. Let $\mathfrak{C}(x) = \sqrt{x}$ $0 \leq x \leq 1$ be a complement function that does not satisfy the monotonic and involutive properties. Then the family of all fuzzy $\mathfrak{C}$- τ_i - closed sets are given by $\mathfrak{C}(\tau_1) = \{0, \{a_{.3}, b_{.4}, c_{.6}\}, \{a_{.4}, b_{.3}, c_{.8}\}, \{a_{.3}, b_{.3}, c_{.6}\}, \{a_{.4}, b_{.4}, c_{.8}\}, \{a_{.8}, b_{.8}, c_{.8}\}, 1\}$ and $\mathfrak{C}(\tau_2) = \{0, \{a_{.3}, b_{.3}, c_{.4}\}, \{a_{.5}, b_{.5}, c_{.5}\}, \{a_{.7}, b_{.7}, c_{.7}\}, 1\}$. Let $\lambda = \{a_{.1}, b_0, c_{.3}\}$. Then it can be evaluated that τ_2 - S $cl_{\mathfrak{C}}(\lambda) = \{a_{.2}, b_0, c_{.4}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\lambda)) = \{a_{.2}, b_{.2}, c_{.5}\}$. Also $\mathfrak{C}\,\lambda = \{a_{.3}, b_0, c_{.5}\}$. Then it can be calculated that τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda) = \{a_{.3}, b_{.4}, c_{.5}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\lambda)) = \{a_{.3}, b_{.4}, c_{.6}\}$. Therefore $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda) = \{a_{.2}, b_{.2}, c_{.5}\}$. Let $\mu = (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda)$. Then τ_2 - S $cl_{\mathfrak{C}}(\mu) = \{a_{.3}, b_{.3}, c_{.5}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mu)) = \{a_{.3}, b_{.4}, c_{.5}\}$. Now $\mathfrak{C}\,\mu = \{a_{.4}, b_{.4}, c_{.7}\}$. Then it can be calculated that τ_2 - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu) = \{a_{.4}, b_{.5}, c_{.7}\}$, τ_1 - S $cl_{\mathfrak{C}}(\tau_2$ - S $cl_{\mathfrak{C}}(\mathfrak{C}\,\mu)) = \{a_{.5}, b_{.6}, c_{.7}\}$. This implies that $(\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}((\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda)) = \{a_{.3}, b_{.4}, c_{.5}\} \nleq \{a_{.2}, b_{.2}, c_{.5}\} = (\tau_1 - \tau_2)$ - S $Bd_{\mathfrak{C}}(\lambda)$.

5. References

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