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On Orthogonal Bimatrices

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Abstract
Orthogonal bimatrices are studied as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

Keywords: Orthogonal matrix, bimatrix, symmetric bimatrix, skew-symmetric bimatrix, orthogonal bimatrix, skew-orthogonal bimatrix

1. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of nxn complex matrices by $\mathcal{C}_{\text{nxn}}$. For $A \in \mathcal{C}_{\text{nxn}}$, $A^T, A^{-1}, A^{\dagger}$ and $\det(A)$ denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively. If $AA^T = A^T A = I$ then A is an orthogonal matrix, where I is the identity matrix. In this paper we study orthogonal bimatrices as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

2. Preliminaries

Definition 2.1 [3]

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bimatrices) where,

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix} \text{ And } A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

$'\cup'$ is just for the notational convenience (symbol) only.

Definition 2.2 [3]

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be any two m x n bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$$\begin{aligned} D_B = A_B + C_B &= (A_1 \cup A_2) + (C_1 \cup C_2) \\ &= (A_1 + C_1) \cup (A_2 + C_2) \end{aligned}$$

Where $A_1 + C_1$ and $A_2 + C_2$ are the usual addition of matrices.

Definition 2.3 [4]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices, then A_B and C_B are said to be equal (written as $A_B = C_B$) if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. (That is, $A_1 = C_1$ and $A_2 = C_2$).

Definition 2.4 [4]

Given a bimatrix $A_B = A_1 \cup A_2$ and a scalar λ, the product of λ and A_B written as λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11}^1 & \lambda a_{12}^1 & \cdots & \lambda a_{1n}^1 \\ \lambda a_{21}^1 & \lambda a_{22}^1 & \cdots & \lambda a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^1 & \lambda a_{m2}^1 & \cdots & \lambda a_{mn}^1 \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{bmatrix} = \left(\lambda A_1 \cup \lambda A_2\right).$$

That is, each element of A_1 and A_2 are multiplied by λ .

Remark 2.5 [4]

If $A_B = A_1 \cup A_2$ be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B.

Definition 2.6 [3]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ are both *n* x *n* square bimatrices then, the bimatrix multiplication is defined as, $A_B \times C_B = \left(A_1 C_1\right) \cup \left(A_2 C_2\right)$.

Definition 2.7 [3]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *m*x*m* square bimatrix. We define $I_B^{m\times m} = I^{m\times m} \cup I^{m\times m} = I_1^{m\times m} \cup I_2^{m\times m}$ to be the identity bimatrix.

Definition 2.8 [3]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a symmetric bimatrix if the component matrices A_1 and A_2 are symmetric matrices. i.e, $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 2.9 [3]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *m*x*m* square bimatrix i.e, A_1 and A_2 are *m*x*m* square matrices. A skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^T$, where $-A_B^T = -A_1^T \cup -A_2^T$ i.e, the component matrices A_1 and A_2 are skew-symmetric.

3. Orthogonal Bimatrices

Definition 3.1

A bimatrix $A_B = A_1 \cup A_2$ is said to be orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = I_B$ (or) $\left(A_1 A_1^T \cup A_2 A_2^T\right) = \left(A_1^T A_1 \cup A_2^T A_2\right) = I_1 \cup I_2$. (That is, the component matrices of A_B are orthogonal.)

That is, $A_B^T = A_B^{-1}$ (or) $\left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right)$.

Remark 3.2

Let $A_B = A_1 \cup A_2$ be a orthogonal bimatrix. If A_1 and A_2 are square and possess the same order then A_B is called square orthogonal bimatrix, and if A_1 and A_2 are of different orders then A_B is called mixed square orthogonal bimatrix.

Example 3.3

(1) $A_B = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix} \cup \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is a square orthogonal bimatrix.

(2) $A_B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cup \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}$ is a mixed square orthogonal bimatrix.

Theorem 3.4

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be given bimatrices.

(1) If A_B is symmetric bimatrix then A_B^T and λA_B are symmetric bimatrices for an arbitrary λ.

(2) If A_B and B_B are symmetric bimatrices of the same order then $A_B \pm B_B$ are symmetric bimatrices as well.

(3) If A_B is symmetric bimatrix then its Moore-Penrose inverse $A_B^\dagger$ is symmetric bimatrix.

Proof

$A_B = A_1 \cup A_2$ Is symmetric bimatrix, if $A_B^T = A_B$ $\left(i.e, \left(A_1 \cup A_2\right)^T = A_1^T \cup A_2^T = A_1 \cup A_2\right)$.

(1) $\left(A_B^T\right)^T = \left(A_1^T \cup A_2^T\right)^T = \left(A_1 \cup A_2\right)$ and

$$\left(\lambda A_B\right)^T = \left(\lambda A_1 \cup \lambda A_2\right)^T = \lambda A_1^T \cup \lambda A_2^T = \lambda A_1 \cup \lambda A_2 = \lambda\left(A_1 \cup A_2\right)$$
$$= \lambda A_B.$$

(2) $$\left(A_B \pm B_B\right)^T = \left[\left(A_1 \cup A_2\right) \pm \left(B_1 \cup B_2\right)\right]^T$$
$$= \left(A_1 \cup A_2\right)^T \pm \left(B_1 \cup B_2\right)^T$$
$$= \left(A_1 \cup A_2\right) \pm \left(B_1 \cup B_2\right)$$
$$= A_B \pm B_B.$$

(3) $\left(A_B^\dagger\right)^T = \left(A_1^\dagger \cup A_2^\dagger\right)^T = \left(A_1^T \cup A_2^T\right)^\dagger = \left(A_1 \cup A_2\right)^\dagger = A_B^\dagger.$

Theorem 3.5

If $A_B = A_1 \cup A_2$ is symmetric bimatrix, then $A_B^T A_B$ and $A_B + A_B^T$ are symmetric bimatrices.

Proof

Let $A_B = A_1 \cup A_2$ is symmetric bimatrix, if $A_B^T = A_B$ $\left(i.e, \left(A_1 \cup A_2\right)^T = A_1^T \cup A_2^T = A_1 \cup A_2\right)$.

Hence, $$\left(A_B^T A_B\right)^T = \left[\left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right)\right]^T$$
$$= \left(A_1^T A_1 \cup A_2^T A_2\right)^T$$
$$= \left(A_1^T A_1\right) \cup \left(A_2^T A_2\right)$$
$$= \left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right)$$
$$= A_B^T A_B.$$

And $$\left(A_B + A_B^T\right)^T = \left[\left(A_1 \cup A_2\right) + \left(A_1^T \cup A_2^T\right)\right]^T$$
$$= \left[\left(A_1 + A_1^T\right) \cup \left(A_2 + A_2^T\right)\right]^T$$
$$= \left(A_1^T + A_1\right) \cup \left(A_2^T + A_2\right)$$
$$= \left(A_1 \cup A_2\right) + \left(A_1^T \cup A_2^T\right)$$
$$= A_B + A_B^T.$$

Theorem 3.6

(1) If $A_B = A_1 \cup A_2$ is skew-symmetric bimatrix then $A_B - A_B^T$ is skew-symmetric bimatrix.

(2) If A_B and B_B are skew symmetric bimatrices of the same order then $A_B \pm B_B$ are skew symmetric bimatrices as well.

Proof

Let $A_B = A_1 \cup A_2$ is skew-symmetric bimatrix, if $A_B^T = -A_B$.

$\left(i.e, \left(A_1 \cup A_2\right)^T = A_1^T \cup A_2^T = -A_1 \cup -A_2\right).$

$$\begin{aligned}\text{Hence, } \left(A_B - A_B^T\right)^T &= \left[\left(A_1 \cup A_2\right) - \left(A_1^T \cup A_2^T\right)\right]^T \\ &= \left[\left(A_1 - A_1^T\right) \cup \left(A_2 - A_2^T\right)\right]^T \\ &= \left(A_1 - A_1^T\right)^T \cup \left(A_2 - A_2^T\right)^T \\ &= \left(A_1^T - A_1\right) \cup \left(A_2^T - A_2\right) \\ &= -\left(A_1 - A_1^T\right) \cup -\left(A_2 - A_2^T\right) \\ &= -\left[\left(A_1 \cup A_2\right) - \left(A_1^T \cup A_2^T\right)\right] \\ &= -\left(A_B - A_B^T\right).\end{aligned}$$

(2)
$$\begin{aligned}\left(A_B \pm B_B\right)^T &= \left[\left(A_1 \cup A_2\right) \pm \left(B_1 \cup B_2\right)\right]^T \\ &= \left(A_1 \cup A_2\right)^T \pm \left(B_1 \cup B_2\right)^T \\ &= \left(A_1^T \cup A_2^T\right) \pm \left(B_1^T \cup B_2^T\right) \\ &= \left[-\left(A_1 \cup A_2\right)\right] \pm \left[-\left(B_1 \cup B_2\right)\right] \\ &= -\left[\left(A_1 \cup A_2\right) \pm \left(B_1 \cup B_2\right)\right] \\ &= -\left(A_B \pm B_B\right).\end{aligned}$$

Theorem 3.7

Let $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ be given bimatrices.

(1) If A_B is skew symmetric bimatrix then A_B^T is skew symmetric bimatrix as well.

(2) If A_B is skew symmetric bimatrix then λA_B is skew symmetric bimatrix for an arbitrary λ.

(3) If A_B is skew symmetric bimatrix then its Moore-Penrose inverse $A_B^\dagger$ is skew symmetric bimatrices.

Proof

$A_B = A_1 \cup A_2$ is skew symmetric bimatrix, if $A_B^T = \left(A_1^T \cup A_2^T\right) = -\left(A_1 \cup A_2\right)$.

(1) $\left(A_B^T\right)^T = \left(A_1^T \cup A_2^T\right)^T = \left(A_1^T\right)^T \cup \left(A_2^T\right)^T = \left(A_1 \cup A_2\right) = -\left(A_1^T \cup A_2^T\right) = -A_B^T.$

(2)
$$\begin{aligned}\left(\lambda A_B\right)^T &= \left(\lambda A_1 \cup \lambda A_2\right)^T \\ &= \lambda A_1^T \cup \lambda A_2^T \\ &= \lambda\left(A_1^T \cup A_2^T\right) \\ &= -\lambda\left(A_1 \cup A_2\right) \\ &= -\lambda A_B.\end{aligned}$$

(3) $\left(A_B^\dagger\right)^T = \left(A_1^\dagger \cup A_2^\dagger\right)^T = \left(A_1^T \cup A_2^T\right)^\dagger = -\left(A_1 \cup A_2\right)^\dagger = -A_B^\dagger.$

Corollary 3.8

Every square bimatrix can be written as a sum of symmetric and skew symmetric bimatrices.

Proof

By Theorem 3.5 and 3.6 we have, $\left(A_B + A_B^T\right)$ is symmetric bimatrix and $\left(A_B - A_B^T\right)$ is skew symmetric bimatrix.

Therefore $A_B = \left(A_1 \cup A_2\right) = \frac{1}{2}\left(A_B + A_B^T\right) + \frac{1}{2}\left(A_B - A_B^T\right)$

$$= \frac{1}{2}\left[\left(A_1 \cup A_2\right) + \left(A_1^T \cup A_2^T\right)\right] + \frac{1}{2}\left[\left(A_1 \cup A_2\right) - \left(A_1^T \cup A_2^T\right)\right]$$

$$= \frac{1}{2}\left[\left(A_1 + A_1^T\right) \cup \left(A_2 + A_2^T\right)\right] + \frac{1}{2}\left[\left(A_1 - A_1^T\right) \cup \left(A_2 - A_2^T\right)\right].$$

Definition 3.9

Let $A_B = \left(A_1 \cup A_2\right) \in C_{n\times n}$ and $B_B = \left(B_1 \cup B_2\right) \in C_{n\times n}$ be given bimatrices. A_B is said to be orthogonally bisimilar to B_B, there exists a orthogonal bimatrix $Q_B = \left(Q_1 \cup Q_2\right) \in C_{n\times n}$ such that $A_B = Q_B^T B_B Q_B$.

$\left(i.e, \left(A_1 \cup A_2\right) = \left(Q_1^T \cup Q_2^T\right)\left(B_1 \cup B_2\right)\left(Q_1 \cup Q_2\right) = \left(Q_1^T B_1 Q_1\right) \cup \left(Q_2^T B_2 Q_2\right)\right).$

That is, the components A_1 and A_2 are orthogonally similar to B_1 and B_2.

Remark 3.10

Orthogonal bimatrices also characterized by their inverses. Before discussing this, we need a brief digression on the transpose of a bimatrix A_B. The main properties of transpose which we have seen are $\left(A_B^T\right)^T = A_B$, $\left(A_B B_B\right)^T = B_B^T A_B^T$, $\det\left(A_B B_B\right) = \det\left(A_B\right) = \det\left(B_B\right)$, $\det\left(A_B\right) = \det\left(A_B^T\right)$, and $\det\left(I_B\right) = 1$ (i.e, determinant value of the identity components I_1 and I_2 are 1).

Theorem 3.11

(1) If $A_B = A_1 \cup A_2$ is orthogonal bimatrix then det $A_B = \pm 1$

(2) If $A_B = A_1 \cup A_2$ is orthogonal bimatrix then $-A_B, A_B^T$ and A_B^{-1} are orthogonal bimatrices as well

(3) If $A_B = A_1 \cup A_2$ and $B_B = B_1 \cup B_2$ are orthogonal bimatrices then $A_B B_B$ is also orthogonal bimatrix.

Proof

$A_B = A_1 \cup A_2$ is orthogonal bimatrix, if $A_B^T = A_B^{-1}$ $\left(i.e, \left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right)\right)$.

(1) $1 = \det\left(I_B\right) = \det\left(A_B A_B^T\right)$

$$= \det\left[(A_1 \cup A_2)(A_1^T \cup A_2^T)\right] = \det\ (A_1 \cup A_2)\ \det\ (A_1^T \cup A_2^T)$$

$$= \det\ (A_1 \cup A_2)\ \det\ (A_1 \cup A_2) = \left[\det\ (A_1 \cup A_2)\right]^2.$$

(2) $\left(-A_B\right)^T = \left[-\left(A_1 \cup A_2\right)\right]^T$

$$= \left[-\left(A_1^T \cup A_2^T\right)\right]$$

$$= \left[-(A_1^{-1} \cup A_2^{-1})\right]$$

$$= \left[-(A_1 \cup A_2)\right]^{-1}$$

$$= \left(-A_B\right)^{-1},$$

$$\left(A_B^T\right)^T = \left(A_1^T \cup A_2^T\right)^T = \left(A_1^{-1} \cup A_2^{-1}\right)^T = \left(A_1^T \cup A_2^T\right)^{-1} = \left(A_B^T\right)^{-1}.$$

and

$$\left(A_B^{-1}\right)^T = \left(A_1^{-1} \cup A_2^{-1}\right)^T = \left(A_1^T \cup A_2^T\right)^{-1} = \left(A_1^{-1} \cup A_2^{-1}\right)^{-1} = \left(A_B^{-1}\right)^{-1}.$$

(3) $\left(A_B B_B\right)^T = \left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^T = \left(B_1 \cup B_2\right)^T \left(A_1 \cup A_2\right)^T$

$$= \left(B_1^T \cup B_2^T\right)\left(A_1^T \cup A_2^T\right)$$
$$= \left(B_1^{-1} \cup B_2^{-1}\right)\left(A_1^{-1} \cup A_2^{-1}\right)$$
$$= \left(B_1 \cup B_2\right)^{-1} \left(A_1 \cup A_2\right)^{-1}$$
$$= \left[\left(A_1 \cup A_2\right)\left(B_1 \cup B_2\right)\right]^{-1}$$
$$= \left(A_B B_B\right)^{-1}.$$

Theorem 3.12

If $A_B = A_1 \cup A_2$ is real skew symmetric bimatrix such that $A_B^2 + I_B = O_B$ $\left(i.e, \left(A_1 \cup A_2\right)^2 + \left(I_1 \cup I_2\right) = O_B\right)$ then A_B is orthogonal bimatrix and is of even order.

Proof

$A_B = A_1 \cup A_2$ is real skew symmetric bimatrix, if $A_B^T = -A_B$

$\Rightarrow \left(A_1^T \cup A_2^T\right) = -\left(A_1 \cup A_2\right)$

$\Rightarrow \left(A_1 \cup A_2\right) = -\left(A_1^T \cup A_2^T\right)$

$\Rightarrow \left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right) = -\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)$

$\Rightarrow \left(A_1 \cup A_2\right)^2 = -\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)$

$\Rightarrow O_B - \left(I_1 \cup I_2\right) = -\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)$

$\Rightarrow \left(I_1 \cup I_2\right) = \left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)$

$\Rightarrow I_B = A_B + A_B^T.$

Again $\left|A_1 \cup A_2\right| = \left|A_1^T \cup A_2^T\right|$ and $\left|kA_1 \cup kA_2\right| = k^n \left|A_1 \cup A_2\right|$, where n is the order of A_B.

Since, $A_B^T = (-1) A_B$

$\Rightarrow \left(A_1^T \cup A_2^T\right) = (-1)\left(A_1 \cup A_2\right)$

$\Rightarrow \left|A_1^T \cup A_2^T\right| = (-1)^n \left|A_1 \cup A_2\right|$

$\Rightarrow \left[1-(-1)^n\right]\left|A_1 \cup A_2\right| = 0.$

Hence, either $\left|A_1 \cup A_2\right| = 0$ or $1-(-1)^n = 0,$ that is n is even.

But $\left(A_1 \cup A_2\right)^2 = O_B - \left(I_1 \cup I_2\right) = -\left(I_1 \cup I_2\right)$

$\Rightarrow \left|A_1 \cup A_2\right|^2 = (-1)^n \left|I_1 \cup I_2\right| = (-1)^n \neq 0.$

Hence the only possibility left that A_B is of even order.

Theorem 3.13

(1) If $A_B = A_1 \cup A_2$ is symmetric bimatrix such that $A_B^2 = I_B$ then A_B is orthogonal bimatrix

(2) If $A_B = A_1 \cup A_2$ is a square bimatrix and $A_B - \frac{1}{2} I_B$ and $A_B + \frac{1}{2} I_B$ are orthogonal bimatrices then A_B is skew symmetric bimatrix.

Proof

(1) $A_B = A_1 \cup A_2$ is symmetric bimatrix and $\left(A_1 \cup A_2\right)^2 = I_1 \cup I_2$.

Hence, $I_1 \cup I_2 = \left(A_1 \cup A_2\right)^2 = \left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right) = \left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right).$

Similarly we can prove $\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right) = I_1 \cup I_2$.

Therefore $\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right) = \left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right) = I_1 \cup I_2$.

Hence, $A_B A_B^T = I_B$.

Since $\left(A_B + \frac{1}{2} I_B\right)\left(A_B + \frac{1}{2} I_B\right)^T = I_B$ and $\left(A_B - \frac{1}{2} I_B\right)\left(A_B - \frac{1}{2} I_B\right)^T = I_B$

$$\Rightarrow \left[\left(A_1 \cup A_2\right) + \frac{1}{2}\left(I_1 \cup I_2\right)\right]\left[\left(A_1 \cup A_2\right) + \frac{1}{2}\left(I_1 \cup I_2\right)\right]^T$$

$$= \left[\left(A_1 \cup A_2\right) - \frac{1}{2}\left(I_1 \cup I_2\right)\right]\left[\left(A_1 \cup A_2\right) - \frac{1}{2}\left(I_1 \cup I_2\right)\right]^T$$

$$\Rightarrow \left[\left(A_1 \cup A_2\right) + \left(A_1^T \cup A_2^T\right)\right] = -\left(A_1 \cup A_2\right) - \left(A_1^T \cup A_2^T\right)$$

$$\Rightarrow \left(A_1^T \cup A_2^T\right) = -\left(A_1 \cup A_2\right).$$

$$\Rightarrow A_B^T = -A_B.$$

Theorem 3.14

Let $J_B = J_1 \cup J_2$ be the diagonal bimatrix with entries ± 1 and $A_B = A_1 \cup A_2$ is orthogonal bimatrix then

(a) $J_B A_B = \left(J_1 A_1 \cup J_2 A_2\right)$ is orthogonal bimatrix.

(b) $A_B J_B = \left(A_1 J_1 \cup A_2 J_2\right)$ is orthogonal bimatrix.

(c) $A_B J_B A_B$ is orthogonal bimatrix.

(d) $J_B A_B J_B$ is orthogonal bimatrix.

Proof

$A_B = A_1 \cup A_2$ Is orthogonal bimatrix, if $A_B^T = A_B^{-1}$ $\left(i.e, \left(A_1^T \cup A_2^T\right) = \left(A_1^{-1} \cup A_2^{-1}\right)\right)$.

Hence

(a)
$$\begin{aligned} \left(J_B A_B\right)^T &= \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^T \\ &= \left(A_1 \cup A_2\right)^T \left(J_1 \cup J_2\right)^T \\ &= \left(A_1^{-1} \cup A_2^{-1}\right)\left(J_1^{-1} \cup J_2^{-1}\right) \\ &= \left(A_1 \cup A_2\right)^{-1} \left(J_1 \cup J_2\right)^{-1} \\ &= \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^{-1} \\ &= \left(J_B A_B\right)^{-1}. \end{aligned}$$

(b)
$$\begin{aligned} \left(A_B J_B\right)^T &= \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^T \\ &= \left(J_1 \cup J_2\right)^T \left(A_1 \cup A_2\right)^T \\ &= \left(J_1^{-1} \cup J_2^{-1}\right)\left(A_1^{-1} \cup A_2^{-1}\right) \\ &= \left(J_1 \cup J_2\right)^{-1} \left(A_1 \cup A_2\right)^{-1} \\ &= \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^{-1} \\ &= \left(A_B J_B\right)^{-1}. \end{aligned}$$

(c) $\left(A_B J_B A_B\right)^T = \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^T$

$$= \left(A_1 \cup A_2\right)^T \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^T$$

$$= \left(A_1 \cup A_2\right)^{-1} \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^{-1}$$

$$= \left[\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^{-1}$$

$$= \left(A_B J_B A_B\right)^{-1}.$$

(d) $\left(J_B A_B J_B\right)^T = \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^T$

$$= \left(J_1 \cup J_2\right)^T \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^T$$

$$= \left(J_1 \cup J_2\right)^{-1} \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\right]^{-1}$$

$$= \left[\left(J_1 \cup J_2\right)\left(A_1 \cup A_2\right)\left(J_1 \cup J_2\right)\right]^{-1}$$

$$= \left(J_B A_B J_B\right)^{-1}.$$

Theorem 3.15

If $A_B = A_1 \cup A_2$ is orthogonal bimatrix such that $\left(A_1^T \cup A_2^T\right) = \left(A_1 \cup A_2\right)$ and $\left(A_1^2 \cup A_2^2\right) = \left(I_1 \cup I_2\right)$ then $A_{B1} + A_{B2} = I_B$ and $A_{B1}A_{B2} = \mathrm{O}_B$, where $A_{B1} = \frac{1}{2}\left[\left(I_1 \cup I_2\right) + \left(A_1 \cup A_2\right)\right]$ and $A_{B2} = \frac{1}{2}\left[\left(I_1 \cup I_2\right) - \left(A_1 \cup A_2\right)\right]$ are projection bimatrices.

Proof

$A_B = A_1 \cup A_2$ Is orthogonal bimatrix, $\left(A_1^T \cup A_2^T\right) = \left(A_1 \cup A_2\right)$ and $\left(A_1^2 \cup A_2^2\right) = \left(I_1 \cup I_2\right)$

$\Rightarrow A_{B1} + A_{B2} = \frac{1}{2}\left[\left(I_1 \cup I_2\right) + \left(A_1 \cup A_2\right)\right] + \frac{1}{2}\left[\left(I_1 \cup I_2\right) - \left(A_1 \cup A_2\right)\right]$

$\Rightarrow A_{B1} + A_{B2} = I_B$. And

$\Rightarrow A_{B1}A_{B2} = \frac{1}{4}\left[\left(I_1 \cup I_2\right) + \left(A_1 \cup A_2\right)\right]\left[\left(I_1 \cup I_2\right) - \left(A_1 \cup A_2\right)\right]$

$$= \frac{1}{4}\left[\left(I_1 \cup I_2\right)^2 - \left(A_1 \cup A_2\right)^2\right]$$

$$= \frac{1}{4}\left[\left(I_1 \cup I_2\right) - \left(I_1 \cup I_2\right)\right]$$

$A_{B1}A_{B2} = \mathrm{O}_B.$

Theorem 3.16

Let $I_B = \left(I_1 \cup I_2\right)$ be a given identity bimatrix and let $Q_B = \left(Q_1 \cup Q_2\right) \in C_{n\times n}$ be an orthogonal bimatrix, and set $R_B = \left(R_1 \cup R_2\right) \equiv Q_B\, I_B\, Q_B^T$ and $S_B = \left(S_1 \cup S_2\right) \equiv Q_B\, A_B\, Q_B^T$. Then

(a) $A_B = A_1 \cup A_2 \in C_{n\times n}$ is orthogonal bimatrix if and only if $S_B^T\, R_B\, S_B = R_B$.

(b) $A_B = A_1 \cup A_2 \in C_{n\times n}$ is symmetric bimatrix if and only if $S_B^T\, R_B = R_B S_B$.

(c) $A_B = A_1 \cup A_2 \in C_{n\times n}$ is skew symmetric bimatrix if and only if $S_B^T\, R_B = -R_B S_B$.

Proof

(a) $S_B^T R_B S_B = R_B$

$$\Leftrightarrow \left(S_1^T \cup S_2^T\right)\left(R_1 \cup R_2\right)\left(S_1 \cup S_2\right) = \left(R_1 \cup R_2\right)$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]^T \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$
$$\left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right] = \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1^T \cup Q_2^T\right)\right] \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$
$$\left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right] = \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[Q_1 A_1^T Q_1^T \cup Q_2 A_2^T Q_2^T\right] \left[Q_1 I_1 Q_1^T \cup Q_2 I_2 Q_2^T\right] \left[Q_1 A_1 Q_1^T \cup Q_2 A_2 Q_2^T\right] = \left[Q_1 I_1 Q_1^T \cup Q_2 I_2 Q_2^T\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T Q_1 I_1 Q_1^T Q_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T Q_2 I_2 Q_2^T Q_2 A_2 Q_2^T\right)\right] = \left[\left(Q_1 I_1 Q_1^T\right) \cup \left(Q_2 I_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T I_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2^T I_2 A_2 Q_2^T\right)\right] = \left[\left(Q_1 I_1 Q_1^T\right) \cup \left(Q_2 I_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T A_1 Q_1^T\right) \cup \left(Q_2 A_2^T A_2 Q_2^T\right)\right] = \left[\left(Q_1 I_1 Q_1^T\right) \cup \left(Q_2 I_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left(Q_1 \cup Q_2\right)\left(A_1^T A_1 \cup A_2^T A_2\right)\left(Q_1^T \cup Q_2^T\right) = \left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)$$

$$\Leftrightarrow \left(A_1^T A_1 \cup A_2^T A_2\right) = \left(I_1 \cup I_2\right)$$

$$\Leftrightarrow \left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right) = \left(I_1 \cup I_2\right)$$

$$\Leftrightarrow A_B^T A_B = I_B.$$

(b) $S_B^T R_B = R_B S_B$

$$\Leftrightarrow \left(S_1^T \cup S_2^T\right)\left(R_1 \cup R_2\right) = \left(R_1 \cup R_2\right)\left(S_1 \cup S_2\right)$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]^T \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$
$$= \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right] \left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1^T \cup Q_2^T\right)\right] \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$
$$= \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right] \left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T\right)\right] \left[\left(Q_1 I_1 Q_1^T\right) \cup \left(Q_2 I_2 Q_2^T\right)\right]$$
$$= \left[\left(Q_1 I_1 Q_1^T \cup Q_2 I_2 Q_2^T\right)\right] \cup \left[\left(Q_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T Q_1 I_1 Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T Q_2 I_2 Q_2^T\right)\right]$$
$$= \left[\left(Q_1 I_1 Q_1^T Q_1 A_1 Q_1^T\right)\right] \cup \left[\left(Q_2 I_2 Q_2^T Q_2 A_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T I_1 Q_1^T\right) \cup \left(Q_2 A_2^T I_2 Q_2^T\right)\right] = \left[\left(Q_1 I_1 A_1 Q_1^T\right)\right] \cup \left[\left(Q_2 I_2 A_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T\right)\right] = \left[\left(Q_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left(Q_1 \cup Q_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1^T \cup Q_2^T\right) = \left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)$$

$$\Leftrightarrow \left(A_1^T \cup A_2^T\right) = \left(A_1 \cup A_2\right)$$

$$\Leftrightarrow A_B^T = A_B.$$

(c) $S_B^T R_B = -R_B S_B$

$$\Leftrightarrow \left(S_1^T \cup S_2^T\right)\left(R_1 \cup R_2\right) = -\left(R_1 \cup R_2\right)\left(S_1 \cup S_2\right)$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]^T \left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$= -\left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]\left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1^T \cup Q_2^T\right)\right]\left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$= -\left[\left(Q_1 \cup Q_2\right)\left(I_1 \cup I_2\right)\left(Q_1^T \cup Q_2^T\right)\right]\left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T\right)\right]\left[\left(Q_1 I_1 Q_1^T\right) \cup \left(Q_2 I_2 Q_2^T\right)\right]$$

$$= -\left[\left[\left(Q_1 I_1 Q_1^T \cup Q_2 I_2 Q_2^T\right)\right] \cup \left[\left(Q_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2 Q_2^T\right)\right]\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T Q_1 I_1 Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T Q_2 I_2 Q_2^T\right)\right]$$

$$= -\left[\left[\left(Q_1 I_1 Q_1^T Q_1 A_1 Q_1^T\right)\right] \cup \left[\left(Q_2 I_2 Q_2^T Q_2 A_2 Q_2^T\right)\right]\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T I_1 Q_1^T\right) \cup \left(Q_2 A_2^T I_2 Q_2^T\right)\right] = -\left[\left[\left(Q_1 I_1 A_1 Q_1^T\right)\right] \cup \left[\left(Q_2 I_2 A_2 Q_2^T\right)\right]\right]$$

$$\Leftrightarrow \left[\left(Q_1 A_1^T Q_1^T\right) \cup \left(Q_2 A_2^T Q_2^T\right)\right] = -\left[\left(Q_1 A_1 Q_1^T\right) \cup \left(Q_2 A_2 Q_2^T\right)\right]$$

$$\Leftrightarrow \left[\left(Q_1 \cup Q_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1^T \cup Q_2^T\right)\right] = -\left[\left(Q_1 \cup Q_2\right)\left(A_1 \cup A_2\right)\left(Q_1^T \cup Q_2^T\right)\right]$$

$$\Leftrightarrow \left(A_1^T \cup A_2^T\right) = -\left(A_1 \cup A_2\right)$$

$$\Leftrightarrow A_B^T = -A_B.$$

Theorem 3.17

(a) If $A_B = A_1 \cup A_2$ is non-singular bimatrix, then $A_B = G_B Q_B$, where $Q_B = \left(Q_1 \cup Q_2\right) \in C_{n\times n}$ is complex orthogonal bimatrix, $G_B = \left(G_1 \cup G_2\right) \in C_{n\times n}$ is complex symmetric bimatrix and G_B is a bipolynomial in $A_B A_B^T$.

(b) Suppose $A_B = G_B Q_B$, where $G_B = G_B^T$ and $Q_B Q_B^T = I_B$. Then $G_B^2 = A_B A_B^T$. If G_B commute with Q_B, then A_B commutes with A_B^T. Conversely, if A_B commutes A_B^T and G_B is a bipolynomial in $A_B A_B^T$, then G_B commutes with Q_B.

Proof

(a) If $A_B = G_B Q_B$ for some symmetric bimatrix G_B $\left(i.e, \left(G_1^T \cup G_2^T\right) = \left(G_1 \cup G_2\right)\right)$ and orthogonal bimatirx Q_B, then

$$\begin{aligned} A_B A_B^T &= G_B Q_B Q_B^T G_B^T \\ &= \left(G_1 \cup G_2\right)\left(Q_1 \cup Q_2\right)\left(Q_1^T \cup Q_2^T\right)\left(G_1^T \cup G_2^T\right) \\ &= \left(G_1 Q_1 Q_1^T G_1^T \cup G_2 Q_2 Q_2^T G_2^T\right) \\ &= \left(G_1 G_1^T \cup G_2 G_2^T\right) \\ &= \left(G_1 \cup G_2\right)\left(G_1^T \cup G_2^T\right) \\ &= \left(G_1 \cup G_2\right)^2 \\ &= G_B^2 \end{aligned}$$

So G_B must be a square root of $A_B A_B^T$. Since $A_1 \cup A_2$ is non-singular bimatrix and has non-singular square roots. Among the square roots of $A_B A_B^T$. Then G_B must be symmetric bimatrix. Since $A_B A_B^T$ is symmetric bimatrix and any bipolynomial in a symmetric bimartrix is symmetric bimatrix. Now define $Q_B \equiv G_B^{-1} A_B$ and compute

$$Q_B Q_B^T = \left(Q_1 \cup Q_2\right)\left(Q_1^T \cup Q_2^T\right) = \left[\left(G_1^{-1} \cup G_2^{-1}\right)\left(A_1 \cup A_2\right)\right]\left[\left(G_1^{-1} \cup G_2^{-1}\right)\left(A_1 \cup A_2\right)\right]^T$$
$$= \left(G_1^{-1} \cup G_2^{-1}\right)\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)\left(G_1^T \cup G_2^T\right)^{-1}$$
$$= \left(G_1^{-1} \cup G_2^{-1}\right)\left(G_1 \cup G_2\right)^2\left(G_1^{-1} \cup G_2^{-1}\right)$$
$$= I_1 \cup I_2$$
$$= I_B.$$

(b) If G_B commutes with Q_B then G_B commutes with Q_B^{-1}, which is a bipolynomial in Q_B. Since $Q_B^{-1} = Q_B^T$, we have

$$A_B^T A_B = \left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right)$$
$$= \left(Q_1^T \cup Q_2^T\right)\left(G_1 \cup G_2\right)^2\left(Q_1 \cup Q_2\right)$$
$$= \left(G_1 \cup G_2\right)^2\left(Q_1^T \cup Q_2^T\right)\left(Q_1 \cup Q_2\right)$$
$$= \left(G_1 \cup G_2\right)^2$$
$$= \left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right).$$

Conversely, if $A_B A_B^T = A_B^T A_B$, then

$$G_B^2 = \left(G_1 \cup G_2\right)^2$$
$$= \left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)$$
$$= \left(A_1^T \cup A_2^T\right)\left(A_1 \cup A_2\right)$$
$$= \left[\left(G_1 \cup G_2\right)\left(Q_1 \cup Q_2\right)\right]^T\left[\left(G_1 \cup G_2\right)\left(Q_1 \cup Q_2\right)\right]$$
$$= \left(Q_1^T \cup Q_2^T\right)\left(G_1 \cup G_2\right)^2\left(Q_1 \cup Q_2\right)$$
$$= \left(Q_1^T \cup Q_2^T\right)\left(A_1 \cup A_2\right)\left(A_1^T \cup A_2^T\right)\left(Q_1 \cup Q_2\right)$$
$$= Q_B^T A_B A_B^T Q_B,$$

So $Q_B A_B A_B^T = A_B A_B^T Q_B$.

That is Q_B commute with $A_B A_B^T$. If G_B is a bipolynomial in $A_B A_B^T$, Q_B must also commutes with G_B.

Theorem 3.18

Let $A_B = A_1 \cup A_2 \in C_{n\times n}$. There exists a complex orthogonal bimatrix $Q_B = \left(Q_1 \cup Q_2\right) \in C_{n\times n}$ and complex symmetric bimatrix $G_B = \left(G_1 \cup G_2\right) \in C_{n\times n}$ such that $A_B = G_B Q_B$ if and only if $rank\left(A_B A_B^T\right)^k = rank\left(A_B^T A_B\right)^k$, for $k = 1, 2, \ldots, n$.

Definition 3.19

A matrix $A_B = A_1 \cup A_2 \in C_{n\times n}$ is called skew orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = -I_B$. Equivalently $A_B^T = -A_B^{-1}$ or $A_B^{-1} = A_B^T$. $\left(i.e., \left(A_1^T \cup A_2^T\right) = -\left(A_1^{-1} \cup A_2^{-1}\right) or \left(A_1^{-1} \cup A_2^{-1}\right) = -\left(A_1^T \cup A_2^T\right)\right)$.

Example 3.20

$$A_B = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{bmatrix} \cup \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$ is a skew orthogonal bimatrix.

Theorem 3.21

(1) If $A_B = (A_1 \cup A_2)$ is skew orthogonal bimatrix of odd order then $\det(A_B) = \pm i.$

(2) If $A_B = (A_1 \cup A_2)$ is skew orthogonal bimatrix of even order then $\det(A_B) = \pm 1.$

(3) If $A_B = (A_1 \cup A_2)$ is skew orthogonal bimatrix then $A_B^T, -A_B$ are skew orthogonal bimatrix.

Theorem 3.22

(1) If $A_B = (A_1 \cup A_2), B_B = (B_1 \cup B_2) \in C_{n \times n}$ are skew orthogonal bimatrix then $A_B B_B$ and $B_B A_B$ are skew orthogonal bimatrix.

(2) If $A_B = (A_1 \cup A_2)$ is orthogonal (skew orthogonal) bimatrix then iA_B is skew orthogonal (orthogonal) bimatrix.

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