International Journal of Applied Research

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2015; 1(12): 246-256
www.allresearchjournal.com
Received: 14-09-2015
Accepted: 16-10-2015

Dipti Prajapati
(Mathematics, /C.U. Shah University, Wadhwan)

Sweta Shah
(Mathematics, /C.U. Shah University, Wadhwan)

Pradeep. J. Jha
Research Guide in Mathematics, L.J. Institute of Engineering & Technology

Correspondence
Pradeep. J. Jha
Research Guide in Mathematics, L.J. Institute of Engineering & Technology

Commutative matrices, Eigen values, and graphs of higher order matrices preserving Libra value

Dipti Prajapati, Sweta Shah, Pradeep. J. Jha

Abstract
In this paper we put forward a class of square matrices having both, the row sum and the column sum for each row and column remaining constant and same; we call this constant as Libra value. Any two member matrices of this infinite sub-class show commutative property for multiplication. In addition to this, these matrices along with their inverses exhibit stringent property in context to Eigen values and Eigen vectors.
We have also shown symmetric matrices which follow some additional properties.

Keywords: *Libra value, Class 3 matrices, Commutative Property, Eigen values, Matrix -graphs*
Notations- *Class3, CJ3 (n x n, L (A) =p), Z_L, L(A), CCJ3 (n x n, L(A) = 3p +k), CCJ3S(n x n, L(A) = 3p +k)*

I. Introduction

It has always remained our continued efforts in quest of an infinite class of matrices such that any two member matrices exhibit commutative property for matrix multiplication and finally we settle down in agreement to Commutativity. This has been elaborated in annexture-1. In sequence of the units discussed in this paper, we have introduced an infinite class; we call it class 3 of square matrices which satisfy property P3. As mentioned in earlier papers, it is a characteristic property in the members of the class. It says that for any member matrix of the class the sum of all the elements of each row and each column remains the same real constant. This constant shall be known as 'Libra Value' and will be denoted as L (A) = p where p is a real value. We have given the general form of square matrices of order n x n and then treated cases in class 3 x 3. These matrices form, what we shall call, a class3 and will be denoted as CJ3(n x n, L(A) = p)These matrices show remarkable properties in connection to algebraic structure of matrices.

As a part of this class CJ3 (n x n, L(A) = p), we have a sub-classto this class, shown by general form

CCJ3 (n x n, L(A) = 3p +k) with p and k as real constants. In addition to all structural properties of the class CJ3(n x n, L(A) = p), the members of the class CCJ3 follow commutative property for matrix multiplication. It is this class which gets wide open, entries to the infinite classes observing commutative property for multiplication.

Again looking for symmetric set of matrices, we have a sub-class to the above class

CCJ3(n x n, L(A) = 3p+k), denoted as CCJ3S(n x n, L(A) = 3p +k)which also conforms to all properties as traits inherited from CCJ3. The resultant matrix of the product also observes the pattern of class CCJ3.

In context to Eigen values and Eigen vectors of the matrices of this class, we have universality for all the member matrices. Also, their inverses do not deviate far away from this property and show fair closeness to this nature. Symmetric matrices as a sub –class to this set, add much to the existing properties.

II. Previous Background and Introduction to Class 3

(1)In earlier Papers we have introduced classification of square matrices of order n x n. We give an overview of first two classes and then focus on the central theme of this paper.

A square matrix A = (a_{ij}) of order n x n is said to be a member of class 1, if the algebraic sum of each of its column member remains constant and same. This constant value is called the Libra value of the matrix; denoted as L(A). In our discussion the Libra value plays a very important role in establishing and interpreting basic properties of matrix algebra. We illustrate this by considering a 3 x 3 matrix.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ p-(a+d) & p-(b+e) & p-(c+f) \end{bmatrix} \in CJ1\ (3 \times 3, L(A) = p) \quad (1)$$

All the entries are real entries; L(A) = p is a real constant associated with the sum of column entries.
Next, we define the member matrix, say A, of class 2 as the one in which the sum of all entries of each row remains constant and same. This constant value is called the Libra value of the matrix; denoted as L(A).We illustrate this by considering a 3 x 3 matrix.

$$A = \begin{bmatrix} a & b & p-(a+b) \\ c & d & p-(c+d) \\ e & f & p-(e+f) \end{bmatrix} \in CJ2\ (3 \times 3, L(A) = p) \quad (2)$$

On introducing these two infinite classes of square matrices, we now focus on class 3.
An n x n square matrix is said to be of the class three if it possesses the property that the algebraic sum of each of its row and each of its column remains constant and same real value. This constant, as mentioned in above cases, is called the Libra value of the matrix.
One easier form of order n x n matrix of class 3 that we shall follow in the notes to follow is,

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1\,n-1} & p-(\sum a_{1\,j}) \\ a_{21} & a_{22} & .. & a_{2\,n-1} & p-(\sum a_{2\,j}) \\ .. & .. & .. & .. & .. \\ a_{n-1\,1} & a_{n-1\,2} & .. & a_{n-1\,n-1} & p-(\sum a_{n-1\,j}) \\ p-(\sum a_{i\,1}) & p-(\sum a_{i2}) & .. & p-(\sum a_{in-1}) & -(n-2)p+\sum\sum a_{i\,j} \end{bmatrix} \quad (3)$$

Where the subscripts 'i and j' run over summation notation from 1 to (n-1). The matrix above represents a general form of matrices of class 3; we denote them symbolically as follows. All entries are real values.
A ∈ $CJ3$ (n x n, L(A) = p)
We cite here a few illustrations

$$A = \begin{bmatrix} 2 & 4 & 1 & 1 \\ 3 & -2 & 6 & 1 \\ -1 & 5 & 3 & 1 \\ 4 & 1 & -2 & 5 \end{bmatrix}$$, we write that A∈ CJ3(4 x 4, L(A) = 8)

$$B = \begin{bmatrix} 5 & 4 & -11 \\ -1 & 1 & -2 \\ -6 & -7 & 11 \end{bmatrix}$$, we write that B ∈ CJ3(3 x 3, L(A) = –2)

To deal with for general purpose we shall consider the following form of 3 x 3 matrix of class 3.

$$A = \begin{bmatrix} a & b & p-(a+b) \\ c & d & p-(c+d) \\ p-(a+c) & p-(b+d) & -p+(a+b+c+d) \end{bmatrix} \in CJ3(3\times3, L(A) = p) \quad (4)$$

with all real entries.
This matrix A will be a symmetric one with the only condition that b = c.
Thus, the most general form of a symmetric matrix of class 3 will appear as follows.

$$A = \begin{bmatrix} a & b & p-(a+b) \\ b & d & p-(b+d) \\ p-(a+b) & p-(b+d) & -p+(a+2b+d) \end{bmatrix} \in CJ3(3\times3, L(A) = p) \quad (5)$$

with all real entries. For a given value of 'p' we have infinite selections of three variables 'a', 'b', and 'd' to construct symmetric matrices of class3. This is a very important point; we will discuss properties of symmetric matrices of class3.

(2) Determinant, Zero Class and Results
We consider the general form of class three matrix as quoted in the above result (4).
For this matrix,

$$A = \begin{bmatrix} a & b & p-(a+b) \\ c & d & p-(c+d) \\ p-(a+c) & p-(b+d) & -p+(a+b+c+d) \end{bmatrix} \in CJ3(\ 3 \times 3, L(A) = p)$$

With |A| = p [3(ad—bc) –p(a + d –b —c)] (6)

Some important Results

(2.1) If L(A) = p = 0 then |A|= 0 or else for ad = bc and a +d = b +c

(2.2) In the case of a symmetric matrix [for b = c in the result (4)], with |A| = p [3(ad–b^2) –p (a + d –2b)]

(2.3) Identity matrix $I_{n \times n} = I_n$ is a matrix of the class 3. In particular, $I_{3x3} \in CJ3(3 \times 3, L(A) = p = 1)$

(2.4) The null /zero matrix is also a matrix of class 3. i.e. 0 ∈CJ3(3 x 3, L(A) = p = 0)

(2.5) From result (6) we can derive that the Libra value L(A) = p is also one of the Eigen value of the matrix of class 3. This is also always true for the matrices of class 1 and class 2. In fact, we will prove this and some additional results in the notes to follow.

(2.6) We have, as an infinite sub-class of class 3, Zero class 3. It is a sub-class of all such matrices which are singular in nature. We denote this class as Z_L.

$Z_L \subset CJ3(n \times n, L(A) = 0)$; all the matrices of this class are singular matrices i.e. ether L(A) = p = 0 or ad = bc and a +d = b + c in the standard form given by the result (4)

As a result to this consideration, the null matrix = **0**$\in Z_L \subset CJ3(3 \times 3, L(A) = p = 0)$

(3) Fundamental algebra and Abelian group

In this section we will deal with fundamental properties and claim all of them to hold true for the member matrices of class3 and in addition to that we will show the role of Libra values in the context of each property.

We consider matrices A, B, and C all of class CJ3 with L(A) = p1, L(B) = p2, and L(C) = p3 and the matrices A, B, and C are in standard form. A = (aij), B = (bij) and C =(cij) with all real entries and α and β as real constants.

(3.1) Equality

Let A and B be the two matrices of the class CJ3(n x n) with L(A) = p1 and L(B) = p2.

A = B ⇔ $a_{ij} = b_{ij} \forall$ i and j from N. This condition implies that Libra values of these matrices are same.

i.e. L(A) = p1= p2 = L(B). If two matrices are equal then their Libra values are same; converse of this statement is not true in general.

(3.2) Addition

Let A and B be the two matrices of the class CJ3(n x n) with L(A) = p1 and L(B) = p2.

We denote their addition as A+B resulting into a matrix C and define it as

$A + B = C = \{c_{ij} = a_{ij} + b_{ij} \forall i, j \in N\}$;

C = { (c_{ij}) |C ∈ CJ3(n x n, L (C) = L(A) + L(B)}; i.e. Libra value of the matrix C is the sum of Libravalues of matrices A and B.

(3.3) Multiplication by a scalar

For a matrix A = {(a_{ij})∈ CJ3(n x n, L(A) = p) and scalar $\alpha \in R$, scalar multiplication of the matrix A by α denoted as αA; it is a matrix of the same order as that of matrix A has.

αA = { (α $a_{ij)}$ for all i and j from N.}; also L(αA) = α L(A)

Comment

(3.3.1)Commutativity in real system implies A+B = B+A

(3.3.2)For α = –1, the product αA is denoted as –1A or –A

(3.4) Abelian group

We consider the set CJ3(n x n, L(A) = p, $p \in R$ } and define '+' as a binary operation on the members of the set CJ3. Let A, B, and C be the member matrices of CJ3; and their Libra values L(A) = p1, L(B) = p2,and L(C) = p3. In this context, we have

(3.4.1) Associative Property

We have A + (B + C) = (A + B) + C; this holds true as the entries of matrices are real numbers and Associative property on the real entries of the matrices A, B, and C holds in the set R. Also we note that L(A + (B+C)) = L ((A+B)+ C) = p1 + p2 +p3.

(3.4.2) Existence of additive identity

As per the defining property of class 3 matrices, it is already established and mentioned earlier thatthe null matrix, denoted as **0**, is a member of class 3 and operate in the class 3 as an identity element.

The null matrix 0 ∈ CJ3(n x n, L(0) = 0). We have For A ∈ CJ3(n x n, L(A) = p)

A + 0 = 0 +A = A

(3.4.3) Existence of additive inverse

For A ∈ CJ3(n x n, L(A) = p) ∃ a matrix –A ∈ CJ3(n x n, L(A) = p)

such that A + (–A) = –A + A = 0. The matrices A and –A of the class CJ3(n x n, L(A) = p) are additive inverses of each other. As per well-defined algebraic structure the class 3 of matrices is a group under the binary operation +.
i.e. (CJ3, +) is a group.

(3.4.4) Commutative Property
For the member matrices A and B of class 3; A ∈ CJ3(n x n, L(A) = p1) and B∈ CJ3(n x n, L(A) = p2)
A + B = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = B + A; following commutative property in the set of real numbers.
This is sufficient enough to claim that the class CJ3(n x n, L(A) = p) is an Abelian group under ‘ +’.
i.e. CJ3(n x n, L(A) = p, +) is an Abelian group. (7)

(3.4.5) Class Z_L is a commutative group
Also we note the fact that the infinite class Z_L is a commutative group under matrix addition (+).
The null matrix being the additive identity and for the matrix A ∈ Z_L the matrix –A serves as additive inverse of A and vice-versa. For any A, B ∈ Z_L; Libra value property i.e. L(A) + L(B) = L(B) + L(A)establishes commutativity.

(4) Ring of Class -3 matrices
(4.1) Multiplications of Matrices
As usual, the operation multiplication of matrices of class 3 is carried out following the same routine and procedure. We mention here the salient features in the case of matrices of class 3.
Let A = (a_{ij})∈ CJ3(n x n,L(A) = p1} and B = (b_{jk}) ∈ CJ3(n x n,L(B) = p2} be the two matrices then
Their product denoted as AB is also a matrix, say C = (c_{ik})∈ CJ3(n x n,L(C) = p3}; where
$c_{ik} = \sum_{j=1}^{j=n} a_{ij}\ b_{jk}$ for i = 1 to n; where n ∈ N. In addition to this, we have an important derivation
L(C) = L(A) L(B)
Also, in general the result of product AB ≠ BA but it preserves the property of the product of Libra values.
We have a few important points to mention at this point.
(4.1.1)If A, and B are the members of class 3(n x n) then the resultant matrices, both AB and BA, are also the members of the class 3. The property implies that multiplication of two matrices is a binary operation on the member matrices of the same class.
(4.1.2)Though AB ≠ BA, yet L(A) L(B) = L(B) L(A) as Libra value stands to represent real numbers associated with the property of the class from which the matrices are referred.
(4.1.3)We have an important connectivity for the product of matrices of different classes.
Let A = (a_{ij})∈ CJ1(n x n,L(A) = p1} and B = {(a_{ij})} ∈ CJ3(n x n,L(A) = p2} then the resultant
AB = {(c_{jj})} ∈ CJ1(n x n,L(AB) = p1. p2} also BA = {(c_{ij})} ∈ CJ1(n x n,L(BA) = p1.p2 }
L(A) L(B) = L(B) L(A) = p1. p2
The fact holds true if the matrices are from class 2 and class 3and the product either AB or BA is a matrix of class2. i.e. AB and BA both ∈ CJ2(n x n,L(BA) = p1.p2 = L(AB)}
(4.1.4)In the product if the matrix A or B or both are the members of the class Z_L then the resultant matrix AB is a member of class Z_L. If any one of both the matrices A or B has Libra value zero then also the resultant matrix AB or BA belong to the zero Libra class Z_L (n x n, L(AB) = L (BA) = 0)

(4.2) Associative Property for Multiplication
For the matrices A, B, and C all from class3 (n x n), with L(A) = p1, L(B) = p2, and L(C) = p3
As all the entries participating in the product are real values and associative property holds true on the set R;
We have $a_{ij}(b_{ij}c_{ij}) = (a_{ij}b_{ij})\ c_{ij}$for all values of i and j extracted from the set N.
We have, as a result, established associative property for multiplication A(BC) = (AB)C
Also, L(A). [L(B).L(C)] = [L(A). L(B)]. L(C) i.e p1.(p2.p3) = (p1.p2).p3 ; being real values

(4.3) Distributive Property
For the matrices A, B, and C all from class3 (n x n), with L(A) = p1, L(B) = p2, and L(C) = p3
We have doubly distributive laws as follows.
A (B+C) = AB + AC and (B+C)A =BA + CA
The proof depends on distributive laws which holds true on the set of real numbers.
$a_{ij}(b_{ij}+ c_{ij}) = a_{ij} . b_{ij} + a_{ij} . c_{ij}$ and $(a_{ij}+ b_{ij}). c_{ij} = a_{ij}. c_{ij} + b_{ij}. c_{ij}$
All values appearing in the above equation are real.
In addition to this, Libra values of these matrices A, B, and C also follow distributive laws.
L(A). [L(B) + L(C)] = L(A) . L(B) + L(A) . L(C) and [L(A) + L(B)] . L(C) = L(A). L(C) + L(B). L(C)

(4.4) Ring (CJ3(n x n, p), +, .)
The infinite set under consideration is the class 3, denoted as CJ3(n x n, L(A) = p) where A is a member matrix of class 3 and the Libra value L(A) = p ; p ∈ R**.**We have introduced two algebraic operations ‘+’, and ‘**.**’ on the members of the set.
(4.4.1)As stated in the result (7), the set CJ3(n x n, L(A) = p) under operation ‘+’ constitutes an Abelian group.
(4.4.2)As stated immediate above, in the section (4.1), (4.2), and (4.3), we have defined multiplication on the members of the set CJ3 (n x n, L(A) = p) ; it is a binary operation. Section (4.2) claims associativity for multiplication while (4.3) doubly

distributive laws enjoining with all above settles down the set along with two operations'+' and '.' i.e. the set (CJ3(n x n, L(A) = p, +, .) a ring.
(4.4.3) As the identity matrix I ∈ CJ3(n x n, L(I) =1), we further claim that the set (CJ3(n x n, L(A) = p, +, .)is a ring with unity.
(4.4.4) Considering the properties of the class Z_L, we derive that (CJ3(n x n, L(A) = p, +, .) is a ring with zero divisors. This is attributed to the Libra value property that if L(A) = p = 0 then |A|= 0 and such matrices are members of class $Z_L \subset$ (CJ3(n x n, L(A) = p)

(5) Commutative Property For Matrix Multiplication On An Infinite Sub-Class Of CJ3(n x n, L(A) = p)
At this stage we consider special type of matrices of class 3 which shows commutative property for multiplication. Let us consider a special form of class 3 matrices as follows.

$$A=\begin{bmatrix} p+k & p-1 & p+1 \\ p+1 & p+k-2 & p+1 \\ p-1 & p+3 & p+k-2 \end{bmatrix} \in \textbf{(}\text{CJ3(3 x 3, L(A) = 3p + k) where p and k } \in \text{R.} \qquad (8)$$

We take p_1, p_2, k_1, and k_2 as real values and compute two matrices A and B as follows.

$$\text{Let } A = \begin{bmatrix} p_1+k_1 & p_1-1 & p_1+1 \\ p_1+1 & p_1+k_1-2 & p_1+1 \\ p_1-1 & p_1+3 & p_1+k_1-2 \end{bmatrix} \in \text{(CJ3(3 x 3, L(A) = 3p1 + k1)}$$

$$\text{And } B = \begin{bmatrix} p_2+k_2 & p_2-1 & p_2+1 \\ p_2+1 & p_2+k_2-2 & p_2+1 \\ p_2-1 & p_2+3 & p_2+k_2-2 \end{bmatrix} \in \text{(CJ3(3 x 3, L(A) = 3p2 + k2)}$$

In this case if we find AB = BA (9)

* Simplified values of each positional term are shown in the annexure -1
* This result also holds true for symmetric matrices of the above form.

$$\text{E.G. } A = \begin{bmatrix} p+k & p+1 & p-1 \\ p+1 & p+k-2 & p+1 \\ p-1 & p+1 & p+k \end{bmatrix} \in \text{(CJ3(3 x 3, L(A) = 3p + k)} \qquad (10)$$

This is a symmetric matrix along both leading and non-leading diagonals.
Illustrations in both cases are given in annexture-2
This general form of matrices given by the relation (8), as long as we know, is long awaited and it will prove a break-through. In the next papers to follow in a very short period, we have found another class, different from this one, is such that its member matrices also observe commutative property for multiplication operation on them.

(6) Some Classical Properties of Sub-Class
(6.1) The matrix structure given by relation (9) is a very typical one and enjoins with it amazing properties of matrix algebra; we show some of them as follows.

$$A=\begin{vmatrix} p+k & p-1 & p+1 \\ p+1 & p+k-2 & p+1 \\ p-1 & p+3 & p+k-2 \end{vmatrix} \in \text{(CJ3(3x 3, L(A) = 3p + k) where p and k} \in \text{R.} \qquad (11)$$

(6.1.1)|A|= det. A = (3p + k) (k-1) (k - 3) and [|A| = 0 for k =1, k = 3, and k = –3p]
(6.1.2)Eigen values are 3p + k, k – 1, and k–3 for any real values ofp and k from R.
(6.1.3)Corresponding to each Eigen values Eigen vectors are [111]', [-101]', and [-1/2-1/21]'
(6.1.4)We prove that Libra value = L(A) of a given matrix is always one of the Eigen values of the matrix under consideration. The proof is given in the form of a theorem in section (6.1.8) below.
(6.1.5) If |A|≠ 0 then A is a non-singular matrix and hence A^{-1} exists.

$$A^{-1} = \frac{1}{(3p+k)(k-1)(k-3)} \begin{bmatrix} (k-4)(k+2p)+1 & -(pk-7p-k-1) & -(p+1)(k-1) \\ -(p+1)(k-1) & (k-1)(k-1+2p) & -(p+1)(k-1) \\ -(pk-7p-k-1) & -5p+k(3+p)-1 & (k-1)(k-1+2p) \end{bmatrix}$$

$A^{-1} \in$ (CJ3(3 x 3, L(A) =1/ (3p + k)) where p and k ∈ R and p ≠ –k/3
(6.1.6) Eigen values of A^{-1} are 1/(3p + k), 1/(k – 1), and 1/(k–3) for real values [for k ≠ 1, k ≠ 3,and k ≠ –3p] of p and k from R.
(6.1.7) Corresponding to each Eigen values Eigen vectors are [1 1 1]', [-1 0 1]', and [-1/2 -1/2 1]'
(6.1.8) Theorem: To prove that Libra value = L(A) of a given matrix is always one of the Eigen values of the matrix under consideration.

Proof
This section we prove that Libra value of a given matrix is one of its Eigen value.
Let us consider the general form of class3 matrix given by relation (3) as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1\,n-1} & p-(\sum a_{1\,j}) \\ a_{21} & a_{22} & .. & a_{2\,n-1} & p-(\sum a_{2\,j}) \\ .. & .. & .. & .. & .. \\ a_{n-1\,1} & a_{n-1\,2} & .. & a_{n-1\,n-1} & p-(\sum a_{n-1\,j}) \\ p-(\sum a_{i\,1}) & p-(\sum a_{i2}) & ..p-(\sum a_{in-1}) & & -(n-2)p+\sum\sum a_{i\,j} \end{bmatrix}$$

Where the subscripts ' i and j ' run over summation notation from 1 to (n-1).
The matrix above represents a general form of matrices of class 3; we denote them symbolically as follows. All entries are real values.
Let A be a matrix from CJ3(n x n, L(A) = p).
Let λ_1 be one of its real Eigen value and $X_1 \neq \mathbf{0}$ be the corresponding Eigen vector.
∴ We have $AX_1 = \lambda_1 X_1$
∴ $(A - \lambda_1 I)\, X_1 = \mathbf{0}$
This gives rise to a system of n equations in n variables.
E.G. $(a_{11} - \lambda_1)x_1 + a_{12}\, x_2 + \ldots\ldots\ldots\ldots + (p - (\sum a_{1\,j}))x_n = 0$
The n^{th} equation is
$(p - (\sum a_{i\,1}))\, x_1 + (p - (\sum a_{i2}))x_2 + \ldots\ldots\ldots + -(n-2)p + \sum\sum a_{i\,j} - \lambda_1) = 0$
These n equations can be written as follows.
$(a_{11})x_1 + a_{12}\, x_2 + \ldots\ldots\ldots\ldots + (p - (\sum a_{1\,j}))x_n = \lambda_1 x_1$
… .. .. … .. ..
.. ..
$(a_{n-1\,1})x_1 + a_{n-1\,2}\, x_2 + \ldots\ldots\ldots\ldots + a_{n-1\,n-1}.x_{n-1} + (p - (\sum a_{n-1\,j}))x_n = \lambda_1 x_{n-1}$
$(p - (\sum a_{i\,1}))\, x_1 + (p - (\sum a_{i2}))x_2 + \ldots\ldots\ldots + (-(n-2)p + \sum\sum a_{i\,j} - \lambda_1)x_n = \lambda_1 x_n$

Adding all these n equations and arranging them
$(a_{11} + a_{21} + \ldots\ldots a_{n-1\,1} + (p - (\sum a_{i\,1})))x_1 + (a_{12} + a_{22} + \ldots\ldots a_{n-1\,2} + (p - (\sum a_{i\,2})))x_2$
$+ \ldots\ldots\ldots\ldots + (p - (\sum a_{1\,j}) + \ldots\ldots\ldots + p - (\sum a_{n-1\,j}) + \{ -(n-2)p + \sum\sum a_{i\,j} - \lambda_1)\, x_n$
R.H.S.= $(\lambda_1)\,(x_1 + x_2 + x_3 + \ldots\ldots + x_n)$

In the expression on left side each bracketed expression yields the Libra value as the basic property of the Libra value.(The sum of all column elements remains constant = p = L(A))
Simplifying this expression we get

$p\,(x_1 + x_2 + x_3 + \ldots\ldots + x_n) = (\lambda_1)\,(x_1 + x_2 + x_3 + \ldots\ldots + x_n)$

This implies $p\,X = \lambda_1 X$; but we have, by definition $AX = \lambda_1\, X$ as $X \neq 0$,
As a result we have $p = \lambda_1$
Hence we conclude that for the matrices of class3 Libra values is one ofthe Eigen values.

(6.2) Properties of the special symmetric matrix of class 3:
Considering the symmetric matrix of class 3 given by the relation (11) above,

$$A = \begin{bmatrix} p+k & p+1 & p-1 \\ p+1 & p+k-2 & p+1 \\ p-1 & p+1 & p+k \end{bmatrix} \in \text{(CJ3(3 x 3, L(A) = 3p + k)}$$

(6.2.1)|A|= det. A = (3p + k)(k+1)(k - 3) and [|A| = 0 for k = –1, k = 3, and k = –3p]
(6.2.2)Eigen values are 3p + k, k + 1, and k–3 for any real values of p and k from R.
(6.2.3)Corresponding to each Eigen values Eigen vectors are [111]', [-101]', and [1-21]'
(6.2.4)We have already proved that Libra value = L(A) of a given matrix is always one of the Eigen values of the matrix under consideration.

(6.2.5)If |A|≠ 0 then A is a non-singular matrix and hence A^{-1} exists.

$$A^{-1} = \frac{1}{(3p+k)(k+1)(k-3)}\begin{bmatrix}(2p+k)(k-2)-1 & -(k+1)(p+1) & 5p-1-pk+k \\ -(k+1)(p+1) & k+2p-1 & -(k+1)(p+1) \\ 5p-1-pk+k & -(k+1)(p+1) & (2p+k)(k-2)-1\end{bmatrix}$$

A^{-1}∈ (CJ3(3 x 3, L(A) =1/ (3p + k)) where p and k ∈ R and p ≠ –k/3
(6.2.6)Eigen values of A^{-1} are 1/(3p + k), 1/(k + 1), and 1/(k–3) for real values [for k ≠ –1, k ≠ 3, and k ≠ –3p] of p and k from R.
(6.2.7)Corresponding to each Eigen values Eigen vectors are [111]', [-101]', and [1-21]'

IIIMATRIX—A Set of Polynomials and Graphs

In this section, we **r**epresent each column of a given matrix (order m x n) in a well-defined pattern of an m^{th} degree polynomial in one variable; say x. This, in turn helps identify each column as an m^{th} degree curve for each column. This convention casts the given matrix of order m x n matrix as a set of n curves each of order m in a single variable.

(1) Matrix—Set of Curves

As per the above convention, we write a column matrix of order m x 1 in the following pattern.

Let **Y** = $\begin{bmatrix}\boldsymbol{a_0}\\ \boldsymbol{a_1}\\ \cdot\\ \cdot\\ \boldsymbol{a_m}\end{bmatrix}$ which is represented as Y = $\sum_{i=0}^{i=m} a_i x^i$ (12)

Thus a m x n matrix is [Y_1 Y_2 …… Y_n] and each one of the n column vector is an m^{th} degree polynomial in real variable x. In addition, in context to reference to follow, we consider each one of the coefficients (a_i) as real variable.

E.G. A_{3x3} matrix A = [Y_1 Y_2 Y_3] where Y_1= $\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}$, Y_2 = $\begin{bmatrix}b_0\\ b_1\\ b_2\end{bmatrix}$, Y_3 = $\begin{bmatrix}c_0\\ c_1\\ c_2\end{bmatrix}$

Where $Y_1 = a_0 + a_1x + a_2 x^2 Y_2 = b_0 + b_1x + b_2 x^2 Y_3 = c_0 + c_1x + c_2 x^2$; each one shows quadratic curve with real coefficients.

(2) Class3 matrix and Graphs

As discussed above paragraph and previous sections, a matrix for which if the sum of all column entries remains constant and same for all columns then we call it a class one matrix. The constant value is called the Libra value of the matrix denoted as L(A).

A = [Y_1 Y_2 Y_3] where Y_1= $\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix}$, Y_2 = $\begin{bmatrix}b_0\\ b_1\\ b_2\end{bmatrix}$, Y_3 = $\begin{bmatrix}c_0\\ c_1\\ c_2\end{bmatrix}$

If$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = c_0 + c_1 + c_2$=
$a_0 + b_0 + c_0 = a_1 + b_1 + c_1 = a_2 + b_2 + c_2$ = constant = p = L(A) then A ∈ CJ3(3 x 3, L(A) = p); p ∈ R
Also, by the convention, each column is a quadratic curve,
$Y_1 = a_0 + a_1x + a_2 x^2$, $Y_2 = b_0 + b_1x + b_2 x^2$, $Y_3 = c_0 + c_1x + c_2 x^2$.
Combining both the facts for x = 1,
We have $Y_1 = a_0 + a_1 + a_2$, $Y_2 = b_0 + b_1 + b_2$, $Y_3 = c_0 + c_1 + c_2$; if the underlying matrix is of class threethen$Y_1 = Y_2 = Y_3$ = P = L(A) = Libra Value of the matrix. The fact conveys that all these quadratic curves are concurrent at the point (1, L(A)= P)

Consider the matrix A=$\begin{bmatrix}5 & 4 & 2\\ 4 & 3 & 4\\ 2 & 4 & 5\end{bmatrix}$ ∈CJ3(3 x 3, L(A) =11) (13)

Corresponding to this matrix the curves are as follows.
$y1=5+4x+2x^2 y2=4+3x+4x^2 y3=2+4x+5x^2$for the curves when x = 1, y = 11.
All these curves intersect at the point (x, y)= (1,11)
The corresponding graphs are as follows.

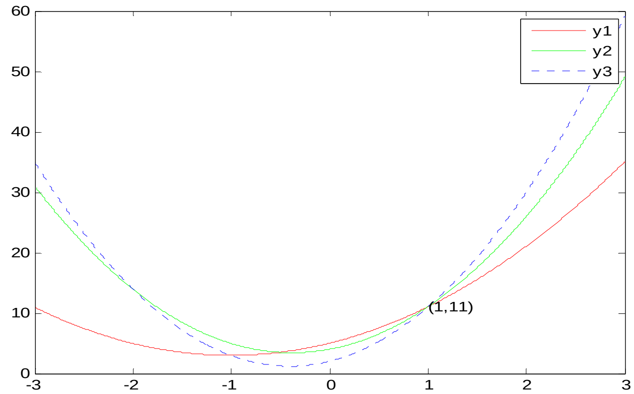

Fig 1

We integrate the set of curves to get corresponding primitive curves.
$y1=5+4x+2x^2, y2=4+3x+4x^2, y3=2+4x+5x^2$
On integration we get,
$y11 = 5x + 2x^2 + (2/3)x^3 + C_{11}$, $y21 = 4x + (3/2)x^2 + (4/3)x^3 + C_{21}$, $y31 = 2x + 2x^2 + (5/3)x^3 + C_{31}$
Where C_{11}, C_{21}, and C_{31} are constants of integration.
We introduce the boundary condition that when x = 1, y = 11
This in turn gives $C_{11}=10/3$, $C_{21}=25/6$, $C_{31}=16/3$
Corresponding to this, primitive curves are,
$y11 = 5x + 2x^2 + (2/3)x^3 + 10/3$, $y21 = 4x + (3/2)x^2 + (4/3)x^3 + 25/6$, $y31 = 2x + 2x^2 + (5/3)x^3 + 16/3$
We express the given curves in matrix notation; denote this as A11;

$$\therefore A11 = \begin{pmatrix} \frac{10}{3} & \frac{25}{6} & \frac{16}{3} \\ 5 & 4 & 2 \\ 2 & \frac{3}{2} & 2 \\ \frac{2}{3} & \frac{4}{3} & \frac{5}{3} \end{pmatrix} \in CJ1(4 \times 3, L(A11) = 11)$$

We introduce one more column to this matrix so that it assumes the form of class3 matrix – having Libra value =L(A11) = 11

$$A11 = \begin{bmatrix} 10/3 & 25/6 & 16/3 & -11/6 \\ 5 & 4 & 2 & 0 \\ 2 & 3/2 & 2 & 11/2 \\ 2/3 & 4/3 & 5/3 & 22/3 \end{bmatrix} \in CJ3(4 \times 4, L(A11) = 11) \quad (14)$$

We follow the same convention of treating each column as an algebraic equation.
They are as follows.
$y11 = 10/3 + 5x + 2x^2 + (2/3)x^3$, $y21 = 25/6 + 4x + (3/2)x^2 + (4/3)x^3$, $y31 = 16/3 + 2x + 2x^2 + (5/3)x^3$
and the new equation of the curve corresponding to the last column is $y41 = -11/6 + (11/2)x^2 + (22/3)x^3$
For all these curves we have a point of graphical concurrence = (1, 11)
We graph the column equations as follows.

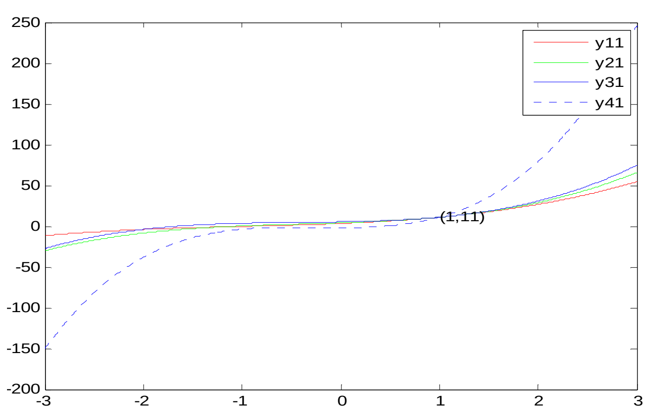

Fig 2

We follow the same procedure of Integrating above equations
y11 = 10/3 + 5x + $2x^2$ + (2/3) x^3, y21 = 25/6 +4x + (3/2)x^2 + (4/3)x^3, y31 = 16/3 +2x + $2x^2$+ (5/3)x^3
andy41= -11/6 + (11/2)x^2 + (22/3)x^3
On Integrating we get,

y12 = (10/3)x +(5/2)x^2 +(2/3)x^3 +(1/6)x^4 + C_{12}
y22 = (25/6)x +2x^2 + (1/2)x3 + (1/3)x^4 + C_{22}
y32 = (16/3)x + x^2 +(2/3)x^3+ (5/12)x^4+ C_{32}
y42 = (-11/6)x + (11/6)x^3 + (11/6)x^4 + C_{42}

Introducing the boundary condition that when x = 1, y = 11; we get

C_{12}= 13/3, C_{22} = 4, C_{32} = 43/12, C_{42} = 55/6

Introducing these constants we have an extended system. We introduce in the matrix form as follows.

$$A2 = \begin{bmatrix} 13/3 & 4 & 43/12 & 55/6 \\ 10/3 & 25/6 & 16/3 & -11/6 \\ 5/2 & 2 & 1 & 0 \\ 2/3 & 1/2 & 2/3 & 11/6 \\ 1/6 & 1/3 & 5/12 & 11/6 \end{bmatrix} \in CJ1(5\text{ x}4, L(A2) = 11)$$

In this matrix we introduce the fifth column to make Libra value L(A2) =11, we have following matrix and we have now 5x5 matrix.

$$A3 = \begin{bmatrix} 13/3 & 4 & 43/12 & 55/6 & -121/12 \\ 10/3 & 25/6 & 16/3 & -11/6 & 0 \\ 5/2 & 2 & 1 & 0 & 11/2 \\ 2/3 & 1/2 & 2/3 & 11/6 & 22/3 \\ 1/6 & 1/3 & 5/12 & 11/6 & 33/4 \end{bmatrix} \in CJ3(5\text{ x}5, L(A3) = 11) \quad (15)$$

If we graph column equations then the corresponding graph is as follows. It is important to note the fact that all these graphs are concurrent at the point (1, 11)

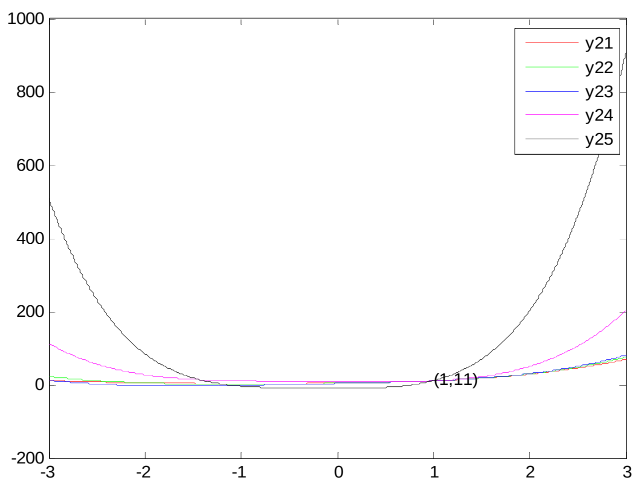

Fig 3

We can continue in the same way and get a sequence of square matrices of the same class. All matrices have the same Libra value and corresponding graphs are concurrent at the same point (1, L(An)) for n = 3,4,5,….

IV Annexure

Annexure-1

We consider the matrices A and B given by relation (9) and both being conformable for matrix multiplication we continue to perform their product.

Let $A = \begin{pmatrix} p1+k1 & p1-1 & p1+1 \\ p1+1 & p1+k1-2 & p1+1 \\ p1-1 & p1+3 & p1+k1-2 \end{pmatrix}$ and $B = \begin{pmatrix} p2+k2 & p2-1 & p2+1 \\ p2+1 & p2+k2-2 & p2+1 \\ p2-1 & p2+3 & p2+k2-2 \end{pmatrix}$

Their product denoted as AB is as follows.

$$AB=\begin{pmatrix} 3p1p2+k1p2+k2p1+k1k2-2 & 3p1p2+k1p2+k2p1-k1-k2+5 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+k2p1+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+p1k2+p2k1-k1-k2+5 & 3p1p2+p1k2+p2k1+3k1+3k2-11 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 \end{pmatrix}$$

And

$$BA=\begin{pmatrix} 3p1p2+k1p2+k2p1+k1k2-2 & 3p1p2+k1p2+k2p1-k1-k2+5 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+k2p1+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+p1k2+p2k1-k1-k2+5 & 3p1p2+p1k2+p2k1+3k1+3k2-11 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 \end{pmatrix}$$

This helps claim that AB = BA
To sound this we take an illustration.
We generate two matrices A and B corresponding to different values; p1 = 1, k1 =2 and p2 = 2 and k2 = 1.
They are as follows.

$$A=\begin{pmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix} B=\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix}$$

Both are conformable for matrix multiplication. Their product yields

$$AB=\begin{pmatrix} 11 & 13 & 11 \\ 11 & 13 & 11 \\ 13 & 9 & 13 \end{pmatrix} \text{and} BA=\begin{pmatrix} 11 & 13 & 11 \\ 11 & 13 & 11 \\ 13 & 9 & 13 \end{pmatrix}$$

Annexture-2
In this annexure we consider a symmetric form given by relation (11)

$$A=\begin{pmatrix} p1+k1 & p1+1 & p1-1 \\ p1+1 & p1+k1-2 & p1+1 \\ p1-1 & p1+1 & p1+k1 \end{pmatrix} \text{and } B=\begin{pmatrix} p2+k2 & p2+1 & p2-1 \\ p2+1 & p2+k2-2 & p2+1 \\ p2-1 & p2+1 & p2+k2 \end{pmatrix}$$

Their product AB and BA is given as follows.

$$AB=\begin{pmatrix} 3p1p2+k1p2+k2p1+k1k2+2 & 3p1p2+k1p2+k2p1+k1+k2-3 & 3p1p2+k1p2+k2p1-k1-k2+1 \\ 3p1p2+k2p1+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+p1k2+p2k1-k1-k2+1 & 3p1p2+p1k2+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2+2 \end{pmatrix}$$

And

$$BA=\begin{pmatrix} 3p1p2+k1p2+k2p1+k1k2+2 & 3p1p2+k1p2+k2p1+k1+k2-3 & 3p1p2+k1p2+k2p1-k1-k2+1 \\ 3p1p2+k2p1+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2-2k1-2k2+6 & 3p1p2+k1p2+k2p1+k1+k2-3 \\ 3p1p2+p1k2+p2k1-k1-k2+1 & 3p1p2+p1k2+p2k1+k1+k2-3 & 3p1p2+k1p2+k2p1+k1k2+2 \end{pmatrix}$$

This helps claim that AB = BA
To sound this we take an illustration.
We generate two matrices A and B corresponding to different values; p1 = 3, k1 = –4 and p2 = 5 and k 2 = –3.
They are as follows.

$$A=\begin{pmatrix} -1 & 4 & 2 \\ 4 & -3 & 4 \\ 2 & 4 & -1 \end{pmatrix} and \text{ } B=\begin{pmatrix} 2 & 6 & 4 \\ 6 & 0 & 6 \\ 4 & 6 & 2 \end{pmatrix}$$

Their product, $AB=\begin{pmatrix} 30 & 6 & 24 \\ 6 & 48 & 6 \\ 24 & 6 & 30 \end{pmatrix}$ and $BA=\begin{pmatrix} 30 & 6 & 24 \\ 6 & 48 & 6 \\ 24 & 6 & 30 \end{pmatrix}$

V.Conclusion
The discussion in the sections above drives us to conclude two important points in the field of matrices. The matrices of class3 have many dominant properties over matrices of class1 and class2. To the best of our views and academic reach the following two of them are the feathers in the global perceptions.

(1) The first one claiming about the Libra value of a matrix of a given class being numerically equal to one of itsCharacteristic roots
(2) The second one concerning related to commutative property of matrix multiplication; i.e. AB = BA

Vision

The next in this sequence we perceive is about verification of right away from establishing the general structure to verification of all algebraic features of matrices of class 4 (A) and class 4(.B).

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