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Vague Congruence's on a Lattice

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Abstract
We study the relationship between vague ideals and vague congruence's on a distributive lattice. And we prove that the lattice of vague ideals is isomorphic to the lattice of vague congruence's on a generalized Boolean algebra.
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Introduction

Artificial intelligence (AI) is a newly developed and highly comprehensive frontier science with rapid development, and it mainly studies how to simulate human intelligence behavior by machine. Intelligent information processing is an important direction in the field of AI. Classical mathematical logic is enough to deal with reasoning in traditional mathematical theory, while, in the real world, there are large numbers of problems with uncertainties. To simulate human intelligence behavior better, uncertainty reasoning becomes a key part of computational science and artificial intelligence, and its rationality should be based on foundations of a kind of scientific logic, which is called nonclassical logic [5, 31, 32]. Therefore, nonclassical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertain information. Many-valued logic is an extension and development of classical logic and has always been a crucial direction in nonclassical logic. As an important many-valued logic, lattice-valued logic [37] has two prominent roles. One is to extend the chain-type truth-valued field of the current logics to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by the general lattice can more effectively reflect the uncertainty of human beings thinking, judging, and decision. Hence, lattice-valued logic has become an important research field and strongly influenced the development of algebraic logic, computer science, and artificial intelligent technology. In 1965, Zadeh introduced the concept of fuzzy set [44]. So far, this idea has been applied to other algebraic structures such as groups, semi groups, rings, modules, vector spaces and topologies and widely used in many fields. Meanwhile, the deficiency of fuzzy sets is also attracting the researcher's attention. For example, a fuzzy set is a single function, and it cannot express the evidence of supporting and opposing. For this reason, the concept of vague set [13] was introduced in 1993 by Gau and Burhrer. In a vague set A, there are two membership function: a truth membership function t_A and a false membership function f_A, where t_A(x) is a lower bound of the grade of membership of x derived from the "evidence for x" and f_A(x) is a lower bound on the negation of x derived from the "evidence against x" and $t_A(x)+ f_A(x)\leq 1$. Thus the grade of membership in a vague set A is a subinterval $[t_A(x), 1-f_A(x)]$ of [0, 1]. The idea of vague sets is an extension of fuzzy sets so that the membership of every element can be divided into two aspects including supporting and opposing. In fact, the idea of vague set is the same with the idea of intuitionistic fuzzy set [1]; so, the vague set is equivalent to intuitionistic fuzzy set. With the development of vague set theory, some structures of algebras corresponding to vague set have been studied. Biswas [6] initiated the study of vague algebras by studying vague groups. Eswarlal [12] studied the vague ideals and normal vague ideals in semi rings. Kham et.al. [26] studied the vague relation and its properties, and moreover intuitionistic fuzzy filters and intuitionistic fuzzy congruence's in a resituated lattice were researched [33, 40, 11, 14, 15, 28]. In this paper, we introduce the concept of vague congruence's on a lattice, and we discuss the relationship between vague ideals and vague congruence's on a distributive lattice. And we prove that the lattice of vague ideals is isomorphic to the lattice of vague congruence's on a generalized Boolean algebra. Finally, we obtain a necessary and sufficient condition for a vague ideal on the direct sum of lattices to be representable as a direct sum of vague ideals on each lattice

2. Preliminaries

Throughout this paper, we will denote the unit interval [0, 1] as I. And Let L = (L, +,) denotes the Lattice generalized Boolean algebra, Where + and. denotes the sup and the inf, respectively.

Definition 2.1 [2]

An element x∈L is said to be relatively complemented if x is complemented in every [a, b] with a ≤ x ≤ b, i.e., x+y = b for some y∈[a, b] such that xy = a. The Lattice L is said to be relatively complemented if each x∈L is relatively complemented.

Definition 2.2 [2]

A relatively complemented distributive lattice with 0 is a generalized Boolean algebra.

Definition 2.3 [2]

Let L be a generalized Boolean algebra and let x,y∈L, Then we define the difference, x-y and the symmetric difference, x⊕ y, of x and y, respectively as follows:
x-y is the relative complement of xy in the interval [0, x] and x ⊕ y = (x-y)+ (y-x). It is easily seen that:
i) x – y ≤ x.
ii) y + (x-y) = x + y.
iii) y(x-y) = 0.

Remark 2.4 [42]

Let L be a generalized Boolean algebra. Then x + y = x ⊕ y ⊕ xy for any x, y ∈L.

Definition 2.5 [22]

A ring with 1 in which every element is idempotent is called a Boolean ring.

Remark 2.6 [42]

Let R be a Boolean ring. Then
i) R is commutative
ii) a+a = 0 for each a∈R.

Result 2.7 [2]

i) If (L, +, ., 0) is a generalized Boolean algebra, then (L, ⊕, ., 0) is a Boolean ring.
ii) If (R, ⊕, ., 0) is a Boolean ring, then (R, +, ., 0) is a generalized Boolean, where x + y = x ⊕ y ⊕ xy. Moreover, x ⊕ y = (x - y) + (y-x).

Definition 2.8 [13]

A Vague set A in the universe of discourse S is a Pair (t_A, f_A) where t_A : S → [0,1] and f_A : S → [0,1] are mappings (called truth membership function and false membership function respectively) where t_A(x) is a lower bound of the grade of membership of x derived from the evidence for x and f_A(x) is a lower bound on the negation of x derived from the evidence against x and t_A(x) + f_A(x) ≤ 1 ∀x∈ S.

Definition 2.9 [13]

The interval [t_A(x),1- f_A(x)] is called the Vague value of x in A, and it is denoted by V_A(x). That is V_A(x) = [t_A(x),1- f_A(x)].

Definition 2.10 [13]

A Vague set A of S is said to be contained in another Vague set B of S. That is A ⊆ B, if and only if V_A(x) ≤ V_B(x). That is t_A(x)≤ t_B(x) and $1 - f_A$(x)≤ $1 - f_B$(x) ∀x ∈ S.

Definition 2.11 [13]

Two Vague sets A and B of S are equal (i.e) A = B, if and only if A⊆B and B⊆A.
(i.e) V_A(x)≤ V_B(x) and V_B(x)≤ V_A(x) ∀x∈ S, which implies t_A(x) = t_B(x) and $1 - f_A$(x) = $1 - f_B$(x).

Definition 2.12 [13]

The Union of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a Vague set C of S, written as C = A ∪ B, whose truth membership and false membership functions are related to those of A and B by t_C = max{t_A, t_B} and $1 - f_C$ = max {$1 - f_A$, $1 - f_B$ }=1-min{f_A, f_B}.

Definition 2.13 [13]
The Intersection of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a Vague set C of S, written as C = A ∩ B, whose truth membership and false membership functions are related to those of A and B by t_C = min{t_A, t_B} and $1 - f_C$ = min {$1 - f_A$, $1 - f_B$ }=1-max{f_A, f_B}.

Definition 2.14 [13]
A Vague set A of S with t_A(x) = 1 and f_A(x) = 0 ∀x∈ S, is called the unit vague set of S.

Definition 2.15 [13]
A Vague set A of S with t_A(x) = 0 and f_A(x) = 1 ∀x∈ S, is called the zero vague set of S.

Definition 2.16 [13]
Let A be a Vague set of the universe S with truth membership function t_A and false membership function f_A, for α,β ∈ [0,1] with α≤β, the (α, β) cut or Vague cut of the Vague set A is a crisp subset $A_{(\alpha,\beta)}$ of S given by $A_{(\alpha,\beta)}$ = {x∈S: V_A(x)≥ (α, β)},(i.e) $A_{(\alpha,\beta)}$ = {x∈S: t_A(x)≥α and 1-f_A(x)≥ β}

Definition 2.17 [13]
The α-cut, A_α of the Vague set A is the (α,α) cut of A and hence it is given by A_α = { x∈S : t_A(x) ≥ α}.

Definition 2.18 [17]
A Fuzzy subset μ of L is called a Fuzzy Sub lattice of L if
i) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$
ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$ ∀ x,y ∈L

Definition 2.19 [17]
A Fuzzy subset μ of L is called a Fuzzy Sublattice of L if
i) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$
ii) $\mu(x \wedge y) \geq \max\{\mu(x), \mu(y)\}$ ∀ x,y ∈L

3. Vague Congruence's
Definition 3.1
Let X be a set and let R∈IFR(X). Then R is called a Vague equivalence relation (in short, VER) on X if it satisfies the following conditions:
i) R is Vague Reflexive, i.e., for each x∈X, R(x,x) = ($\vee_{y,z\in X} V_R(y,z)$)
ii) R is Vague symmetric, i.e., R(x,y) = R(y,x), for any x,y∈X
iii) R is Vague transitive, i.e., RoR⊆R.
We will denote the set of all VERs on X as VE(X).

Definition 3.2
Let L be a Lattice and let R∈VE(L). Then R is called a Vague congruence (in short, VC) on L if it satisfies the following conditions: for any x_1, y_1, x_2, y_2 ∈L,
i) $V_R(x_1 + x_2,\ y_1 + y_2) \geq V_R(x_1,\ y_1) \wedge V_R(x_2,\ y_2)$
ii) $V_R(x_1 . x_2,\ y_1 . y_2) \geq V_R(x_1,\ y_1) \wedge V_R(x_2,\ y_2)$
We will denote the set of all VCs on L as VC(L).

Definition 3.3
Let A∈VI(L). We define a complex mapping $R_A = V_{R_A}$: L × L → I×I as follows: for any x,y∈L,
V_{R_A}(x,y) = $\vee_{a+x=a+y} V_A$(a). Then R_A is called the Vague relation induced by A.

Lemma 3.4
Let L be a distributive Lattice. If A∈VI(L), then R_A ∈VC(L).
Proof
Let x∈L. Then V_{R_A}(x, x) = $\vee_{a+x=a+x} V_A$(a) = $\vee_{a\in L} V_A$(a). Let y,z∈L. Then V_{R_A}(y, z) = $\vee_{b+xy=b+z} V_A$(b) ≤$\vee_{b\in L} V_A$(b) = V_{R_A}(x, x). Thus V_{R_A}(x, x) ≥ $\vee_{y,z\in L} V_{R_A}$(y,z). So R_A(x, x) = $\vee_{y,z\in L} V_{R_A}$(y,z), i.e., R_A is Vague reflexive. It is clear that R_A is Vague symmetric from the definition of symmetry. Let x,y∈L. Then $V_{R_A \circ R_A}$(x, y) = $\vee_{z\in L}[V_{R_A}$(x, z)∧ V_{R_A}(z, y)] = $\vee_{z\in L}$[($\vee_{a+x=a+z} V_A$(a))∧($\vee_{b+z=b+y} V_A(b)$)]≤ $\vee_{z\in L}$[($\vee_{(a+b)+x=(a+b)+z} V_A$(a+b))∧(($\vee_{(a+b)+z=(a+b)+y} V_A(a+b)$)] = $\vee_{(a+b)+x=(a+b)+y} V_A$(a+b) = V_{R_A}(x, y). Then R_Ao R_A⊂ R_A. So R_A is Vague transitive. Hence R_A∈ VE(L). Now let x_1, y_1, x_2, y_2 ∈L. Suppose b+ x_1 = b+ y_1 and c+x_2= c+y_2. Then (b+ x_1)(c+x_2) = (b+ y_1) (c+y_2). Since L is distributive, (bc+ x_1c+ x_2b)+ $x_1 x_2$ = (bc+y_1c+y_2b)+ $y_1 y_2$. Since (b+ x_1)c = (b+ y_1)c and (c+x_2)b = (c+y_2)b, bc+ x_1c+ x_2b = bc+y_1c+y_2b. Since A∈VI(L), V_A(bc+ x_1c+ x_2b) = V_A(bc)∧ V_A (x_1c)∧ V_A(x_2b)≥ V_A(b)∧ V_A(c). Then $V_{R_A}(x_1,\ y_1) \wedge V_{R_A}(x_2,\ y_2)$ = ($\vee_{b+x_1= b+y_1} V_A(b)$)∧ ($\vee_{c+x_2= c+y_2} V_A(c)$) =

$\vee_{b+x_1=b+y_1, c+x_2=c+y_2}$ $[V_A(b)\wedge V_A(c)] \leq \vee_{(bc+x_1c+x_2b)+x_1x_2=(bc+y_1c+y_2b)+y_1y_2}$ $V_A(bc+x_1c+x_2b) = V_A(x_1x_2, y_1y_2)$. It can be easily seen that $V_R(x_1+x_2, y_1+y_2) \geq V_R(x_1, y_1) \wedge V_R(x_2, y_2)$. Hence $R_A \in$VC(L). This completes the proof.

Definition 3.5

Let R∈VE(L). We define a mapping $A_R = V_{A_R}$: L → I×I as follows; for each x∈L, $V_{A_R}(x) = \wedge_{y\in L} V_R(xy, x)$. Then A_R is called a Vague set in L induced by R.

Lemma 3.6:

Let L be a distributive lattice. If R∈VC(L), then A_R ∈VI(L).

Proof

Let x,y∈L. Then V_{A_R} (x+y)∧ V_{A_R} (xy) = $(\wedge_{z\in L} V_R((x+y)z, x+y)$ ∧$(\wedge_{z\in L} V_R((xy)z, xy)$ = $\wedge_{z\in L}$[V_R((xz+yz, x+y)∧ V_R(x(yz), xy)]≥$\wedge_{z\in L}$[V_R(xz,x)∧ V_R(yz,y)∧ V_R(x,x)∧ $V_R(yz, y)$]= $\wedge_{z\in L}$[V_R(xz,x)∧ V_R(yz,y)] = $(\wedge_{z\in L} V_R(xz, x))$∧ $(\wedge_{z\in L} V_R(yz, y))$ = $V_{A_R}(x)\wedge V_{A_R}(y)$. Thus $A_R \in$VL(L). For any x,y∈L, suppose x≥y. Then $V_{A_R}(x)$ = $\wedge_{z\in L} V_R(xz, x$ = $\wedge_{z\in L} V_R(xyz, xy)$≥$\wedge_{z\in L}$[$V_R(x, x)\wedge V_R$(yz,y)] = $\wedge_{z\in L} V_R(yz, y)$ = $V_{A_R}(y)$. Hence $A_R \in$VI(L).

Theorem 3.7

Let L be a distributive lattice with 0, If A∈VI(L), then A = A_{R_A}.

Proof

Let x∈L. Then $V_{A_{R_A}}(x)$ = $\wedge_{z\in L} V_{R_A}(xz, x)$. Since x+xz = x+x, $V_{R_A}(xz, x)$ = $\vee_{a+xz=a+x} V_A(a) \geq V_A(x)$. Thus $V_{A_{R_A}}(x)$ = $\wedge_{z\in L} V_{R_A}(xz, x) \geq V_A(x)$. So A⊂$A_{R_A}$. On the other hand, $V_{A_{R_A}}(x)$ = $\wedge_{z\in L} V_{R_A}(xz, x) \leq V_{R_A}(x0, x)$ = $\vee_{a+x0=a+x} V_A$(a) = $\vee_{a=a+x} V_A$(a) = $\vee_{a\geq x} V_A(a) \leq V_A(x)$(since A∈vi(L)). So A_{R_A}⊂A. Hence A = A_{R_A}.

Lemma 3.8

Let L be a generalized Boolean algebra and let A∈VI(L). Then A(x⊕y) = $R_A(x, y)$ for any x,y∈L.

Proof

We know that x+(x⊕y) = y+(x⊕y). So, $V_{R_A}(x, y)$ = $\vee_{a+x=a+y} V_A$(a)≥V_A(x⊕y). Now let a∈L with a+x = a+y. Then, x⊕y≤a. Since A∈VI(L), V_A(a)≤ V_A(x⊕y). Thus $V_{R_A}(x, y)$ = $\vee_{a+x=a+y} V_A$(a)≤ $\vee_{a+x=a+y} V_A$(x⊕y) = V_A(x⊕y). Hence A(x⊕y) = $R_A(x, y)$.

Lemma 3.9

Let L be a lattice with 0. If R∈VC(L), then A_R(x) = R(x,0) for each x∈L.

Proof

Let x∈L. Then $V_{A_R}(x)$ = $\wedge_{z\in L} V_R$(xz,x) ≤V_R(x0,x) = V_R(x,0). Now let z∈L. Then V_R(xz,x) = V_R(xz+0,xz+x)≥ V_R(xz,xz)∧ V_R(0,x) = V_R(x,0). Thus $V_{A_R}(x)$ = $\wedge_{z\in L} V_R(xz, x) \geq \wedge_{z\in L} V_R(x, 0)$ = V_R(x,0). Hence A_R(x) = R(x, 0).

Lemma 3.10:

Let A_1, A_2∈VI(L) and let R_1, R_2∈VC(L),

i) If A_2⊂A_1 then R_{A_2}⊂R_{A_1}.

ii) If R_2⊂R_1 then A_{R_2}⊂A_{R_1}.

Proof: Trivial.

Theorem 3.11:

Let L be a generalized Boolean algebra. If R∈VC(L), then R_{A_R} = R.

Proof

For any x,u,v∈L, let x+u = x+v. Then A_R(x) = R(x,0). Thus V_R(x+u, u) = V_R(x+u, 0+u) ≥V_R(x, 0)∧ V_R(u, u)= V_R(x, 0) = V_{A_R}(x). By the similar arguments, we have V_R(x+v, v)≥ V_{A_R}(x). Thus V_R(u, v) ≥ V_R(u, x+u) ∧ V_R(x+u, v)(since R is transitive) = V_R(u, x+u) ∧ V_R(x+v, v) ≥V_{A_R}(x). So V_{A_R}(u, v) = $\vee_{x+u=x+v} V_{A_R}$(x) ≤ $\vee_{x+u=x+v} V_R$(u,v) = V_R(u, v). Hence R_{A_R} ⊆ R. Now let x,y∈L. Then by Lemma 2.8 and 2.9, $V_{R_{A_R}}$(x, y) = V_R(x ⊕ y) = V_R(x⊕y, 0) = V_R((x-y)+(y-x), 0) ≥ V_R(x-y, 0)∧ V_R(y-x, 0)(since R∈VC(L)) = V_R(x(x-y), y(x-y))∧ V_R(y(y-x), x(y-x)) ≥ V_R(x, y) (since R∈VC(L)). Thus R ⊆ R_{A_R} . Hence R_{A_R} = R. It is obvious that the intersection of an arbitrary family of Vis (resp.VCs) is a VI(resp.VC). But this is not true for union. Let A_1, A_2 ∈ VI(L) and we define a VI A_1 + A_2 of L by A_1 + A_2 = ⋂{A∈VI(L) : A_1∪A_2 ⊂ A}. Then clearly, A_1 + A_2 is the smallest VI of L containing A_1 and A_2. If we define a VC R_1 + R_2 by R_1 + R_2 = ⋂{R∈VC(L) : R_1∪R_2 ⊂ R}, R_1 + R_2 is the smallest VC on L containing R_1 and R_2. Hence (VI(L), ∩, +) and (VC(L), ∩, +) are complete lattices.

Theorem 3.12:

Let L be a generalized Boolean algebra. Then (VI(L), ∩, +) ≅ (VC(L), ∩, +).

Proof

We define a mapping f: VC(L) → VI(L) as follows: for each R ∈ VC(L), f(R) = A_R. Then clearly f is well-defined . Let A∈VI(L). Then, R_A∈VC(L). Therefore A_{R_A} = A i.e., f(R_A) = A. Thus f is surjective. For any R_1, R_2 ∈VC(L), suppose f(R_1) = f(R_2). Then A_{R_1} = A_{R_2}. Hence R_1 = $R_{A_{R_1}}$ = $R_{A_{R_2}}$ = R_2. Thus f is injective. So f is bijective. Let R_1, R_2∈VC(L) and let x∈L.

Then $V_{f(R_1 \cap R_2)}(x)$ = $V_{A_{R_1 \cap R_2}}$ (x) = $\wedge_{x\in L} V_{R_1 \cap R_2}$ (xz, x) = $\wedge_{x\in L}$ [V_{R_1} (xz, x)$\wedge V_{R_2}$(xz, x)]= ($\wedge_{x\in L}$ V_{R_1}(xz, x)) $\wedge$ ($\wedge_{x\in L}$ V_{R_2}(xz, x)) = $V_{A_{R_1}}$(x)$\wedge$ $V_{A_{R_2}}$(x) = $V_{f(R_1)}$(x)$\wedge$ $V_{f(R_2)}$(x) = $V_{f(R_1)\cap f(R_2)}(x)$. So f($R_1 \cap R_2$) = f(R_1)$\cap$f(R_2). On the other hand, $R_i \subset R_1 + R_2$ (i=1,2). Therefore $A_{R_i} \subset A_{R_1 + R_2}$ (i=1,2). Then A_{R_1}+ $A_{R_2} \subset A_{R_1 + R_2}$. Thus f(R_1)+f(R_2) $\subset$f($R_1 + R_2$) . Let A$\in$VI(L) such that f(R_i)$\subset$A(i=1,2). Then $A_{R_i}$$\subset$A(i=1,2). Therefore $R_{A_{R_i}} \subset R_A$ (i=1,2). Hence $A_{R_i}$$\subset$ A(i=1,2). Thus $R_i \subset R_A$ (i= 1,2) i.e., $R_1 + R_2$ $\subset R_A$. Therefore $A_{R_1 + R_2} \subset A_{R_A}$ = A. Then f($R_1 + R_2$) $\subset$ f(R_1)+f(R_2). So f($R_1 + R_2$) = f(R_1)+f(R_2). Hence f is an isomorphism. This completes the proof.

4. Products of Vague Ideals

Definition 4.1:

Let A$\in$VS(L_1) and $\in$VS(L_2). We define a complex mapping A×B = $V_{A\times B}$: $L_1 \times L_2 \rightarrow$ I× I as follows : for each (x,y) $\in$ $L_1 \times L_2$, $V_{A\times B}$ = V_A(x)$\wedge$ V_B(y) . Then A×B is called the product of A and B. It is clear that A×B $\in$ VS($L_1 \times L_2$) from definition 3.1.

Definition 4.2

Let A$\in$VS($L_1 \times L_2$) . We define two complex mappings π_1(A) = $V_{\pi_1(A)}$: $L_1 \rightarrow$ I×I and π_2(A) = $V_{\pi_2(A)}$: $L_2 \rightarrow$ I×I. as follows, respectively : π_1(A)(x) = $\vee_{y\in L_2} V_A$(x, y) for each x $\in L_1$ and π_2(A)(y) = $\vee_{x\in L_1} V_A$(x, y) for each y $\in L_2$. Then π_1(A)(x) and π_2(A)(x) are called the projections of A on L_1 and L_2, respectively. It is clear that π_1(A) $\in$VS(L_1) and π_2(A)$\in$VS(L_2).

Proposition 4.3

1. If $A_i \in$VL(L_i) [resp. VI(L_i)] (I = 1,2), then $A_1 \times A_2 \in$VL($L_1 \times L_2$)[resp.VI($L_1 \times L_2$)].
2. If A$\in$VL($L_1 \times L_2$) [resp.VI($L_1 \times L_2$)], then π_i(A)$\in$VL(L_i)[resp.VI(L_i)] (I = 1,2).

Proof

1. The proof is obvious.
2. Suppose A$\in$VL($L_1 \times L_2$) and let x,y$\in L_1$. Then $V_{\pi_1(A)}$ (x+y) = $\vee_{z\in L_2} V_A$(x+y, z) = $\vee_{z_1,z_2 \in L_2} V_A$(x+y, z_1+ z_2)$\geq \vee_{z_1,z_2 \in L_2}$[$V_A(x, z_1)\wedge$ V_A(y, z_2)] = ($\vee_{z_1\in L_2} V_A(x, z_1)$)$\wedge$($\vee_{z_2\in L_2} V_A(y, z_2)$) = $V_{\pi_1(A)}$(x)$\wedge$ $V_{\pi_1(A)}$(y). Also, $V_{\pi_1(A)}$ (xy) = $\vee_{z\in L_2} V_A$(xy, z) = $\vee_{z\in L_2} V_A$(xy, z) = $\vee_{z_1,z_2 \in L_2} V_A$(xy, $z_1 z_2$)$\geq \vee_{z_1,z_2 \in L_2}$[$V_A(x, z_1)\wedge$ V_A(y, z_2)] = ($\vee_{z_1\in L_2} V_A(x, z_1)$)$\wedge$($\vee_{z_2\in L_2} V_A(y, z_2)$) = $V_{\pi_1(A)}$(x)$\wedge$ $V_{\pi_1(A)}$(y). Hence π_1(A) $\in$VL(L_1). Similarly, we can see that π_2(A)$\in$VL(L_2). By the similar arguments, we can show the rest.

Definition 4.4

Let A$\in$VS($L_1 \times L_2$) and let a$\in L_2$, b$\in L_1$. We define two complex mappings ${A_1}^{(a)}$ = V_{A_1}(a) : $L_1 \rightarrow$ I×I and ${A_2}^{(b)}$ = V_{A_2}(b) : $L_2 \rightarrow$ I×I. as follows respectively. ${A_1}^{(a)}$ (x) = V_A(x, a) for each x$\in L_1$ and ${A_2}^{(b)}$ (y) = V_A(b, y) for each y$\in L_2$. It is clear that ${A_1}^{(a)} \in$VS(L_1) and ${A_2}^{(b)} \in$VS(L_2). Then ${A_1}^{(a)}$ and ${A_2}^{(b)}$ called the marginal Vague sets of A (with respect to a and b)

Proposition 4.5

If A$\in$VL($L_1 \times L_2$)[resp.VI($L_1 \times L_2$)], then ${A_1}^{(a)}$ $\in$VL(L_1) [resp.VI(L_1)] for each a$\in L_2$ and ${A_2}^{(a)}$ $\in$VL(L_2) [resp.VI(L_2)] for each b$\in L_1$.

Proof: follows from definitions.

Lemma 4.6

If A$\in$VI($L_1 \times L_2$), then for each a$\in L_2$ and b$\in L_1$, ${A_1}^{(a)} \times {A_2}^{(b)} \subset$ A $\subset \pi_1$(A) × π_2(A).

Proof: Let (x, y)$\in$ $L_1 \times L_2$. Then $V_{{A_1}^{(a)}\times {A_2}^{(b)}}$(x, y) = $V_{{A_1}^{(a)}}(x) \wedge V_{{A_2}^{(b)}}(y)$ = V_A(x, a) $\wedge$ V_A(b, y) ≤ V_A(x+b, a+y) = V_A((x,y) + (b, a)) ≤ V_A(x, y) (since A$\in$VI($L_1 \times L_2$)). Thus ${A_1}^{(a)} \times {A_2}^{(b)} \subset$ A. On the other hand, V_A(x, y)≤$\vee_{z\in L_2} V_A(x, z)$ = $V_{\pi_1(A)}$(x). By the similar arguments, we have V_A(x, y) ≤ $V_{\pi_2(A)}$(y). Thus V_A(x, y) ≤ $V_{\pi_1(A)}$(x) $\wedge V_{\pi_2(A)}$(y) = $V_{\pi_1(A)\times\pi_2(A)}$(x, y). So A $\subset$ $\pi_1(A)\times\pi_2(A)$. This completes the proof.

Theorem 4.7

Let L_1 and L_2 be two lattices with 0 and let A$\in$VI($L_1 \times L_2$). Then A is the product of a VI of L_1 and of a VI of L_2 if and only if ${A_1}^{(0)} \times {A_2}^{(0)} = \pi_1(A)\times\pi_2(A)$.

Proof

Suppose the necessary condition holds. By Lemma 3.6, A = ${A_1}^{(0)} \times {A_2}^{(0)}$. Since A$\in$VI($L_1 \times L_2$), by Proposition 3.5, ${A_1}^{(0)} \in$VI(L_1) and ${A_2}^{(0)} \in$VI(L_2). Hence the sufficient condition holds. Let A is the product of a VI of L_1 and of a VI of L_2, say A = ${A_1}' \times {A_2}'$, where ${A_i}' \in$VI(L_i) (i = 1,2). Then clearly $V_{{A_i}'}(x) \leq V_{{A_i}'}$(0) (i = 1,2). Thus $\vee_{x\in L_i} V_{{A_i}'}(x) = V_{{A_i}'}$(0) (I = 1,2). So $V_{{A_1}^{(0)}}(x)$ = V_A(x, 0) = $V_{{A_1}'}(x) \wedge V_{{A_2}'}(0)$ = $V_{{A_1}'}(x)$ $\wedge$ ($\vee_{y\in L_2} V_{{A_2}'}(y)$) = $\vee_{y\in L_2}[V_{{A_1}'}(x) \wedge V_{{A_2}'}(y)]$ = $\vee_{y\in L_2} V_A$(x, y) = $V_{\pi_1(A)}$(x). Thus ${A_1}^{(0)} = \pi_1(A)$. Similarly, we have ${A_2}^{(0)} = \pi_2(A)$. Hence ${A_1}^{(0)} \times {A_2}^{(0)} = \pi_1(A)\times\pi_2(A)$. This completes the proof.

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