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Stochastic epidemic models with Poisson infection and carrier rates

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Abstract

In this paper, we have derived two stochastic epidemic models one is with Poisson infection and carrier rates another with Poisson infection, carrier and removal rates.

Keywords: Epidemics, susceptible, infection rate, carrier rate, removal rate.

1. Introduction

Mathematical and theoretical biology is an interdisciplinary scientific research field with wide range of applications in biology, medicine and biotechnology. The field may be referred to as mathematical biology or biomathematics to stress the mathematical side, or as theoretical biology to stress the biological side. It includes at least four major subfields: biological mathematical modeling, relational biology/complex systems biology (CSB), bioinformatics and computational bio modeling/bio computing. Mathematical biology aims at the mathematical representation, treatment and modeling of biological processes, using a variety of applied mathematical techniques and tools. It has both theoretical and practical applications in biological, biomedical and biotechnology research. For example, in cell biology, protein interactions are often represented as cartoon models, which, although easy to visualize, do not accurately describe the systems studied. In order to do this, precise mathematical models are required. By describing the systems in a quantitative manner, their behaviour can be better simulated, and hence properties can be predicted that might not be evident to the experimenter. Mathematical biology is a very active and fast growing interdisciplinary area in which mathematical concepts, techniques, and models are applied to a variety of problems in developmental biology and biomedical sciences.

The occurrence of more cases of a disease than would be expected in a community or region during a given time period is called an epidemic. An epidemic is the rapid spread of infectious disease to a large number of people in a given population within a short period of time, usually two weeks or less. An epidemic may be restricted to one location; however, if it spreads to other countries or continents and affects a substantial number of people, it may be termed a pandemic [12].

Epidemiology is one of the most important fields of research. The modeling and analysis of infectious diseases have been done by many authors. The classical theoretical papers on epidemic models by Kermack and Mckendrick [6-8] have had a major influence in the development of mathematical models. The modeling literature is now extensive and growing very quickly. Bailey [1] gave a good introduction to the variety of problems and models for the spread and control of infectious disease. The article by Hethcote [4] reviewed three basic epidemiological models. Diekmann and Heesterbeek [3] presented a concentrated review to this field and they discussed how to use biological assumptions in constructing models and present applications.

In mathematics and physics, a deterministic system is a system in which no randomness is involved in the development of future states of the system [13]. A deterministic model will thus always produce the same output from a given starting condition or initial state [14]. For example, physical laws that are described by differential equations represent deterministic systems, even though the state of the system at a given point in time may be difficult to

describe explicitly. Markov chains and other random walks are not deterministic systems, because their development depends on random choices.

Stochastic means being or having a random variable. A stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more inputs over time. The random variation is usually based on fluctuations observed in historical data for a selected period using standard time series techniques. Distributions of potential outcomes are derived from a large number of simulations which reflect the random variation in the input.

Basic Concepts

The term epidemiology is the study of health-event, health-characteristic, or health- determinant patterns in a society. Now before we go further in this study, here the basic concepts are introduced which will be useful in the following sections.

Definition 1.1 ^[2]

An *epidemic* is an unusually large short – term outbreak of a disease, such as Cholera and Plague etc.

The spread of disease depends on:

- The mode of transmission.
- Susceptibility.
- Infections period.
- Resistance and many other factors.

Definition 1.2 ^[9]

Any group of individuals, usually of a single species, occupying a given area at the same time is known as *population*.

The population in an epidemiological model can be classified into three types: Susceptible, Infected and Removal which are denoted by $S(t)$, $I(t)$ and $R(t)$ respectively, where $S(t)$ represent the number of individuals in the population who can be infected, while $I(t)$ represent the number of infected individuals in the population, that is, those with the disease who are actively transmitting it, $R(t)$ denotes the number of individuals removed from the population by recovery, death, immunization or other means ^[5].

In probability theory, a Poisson process is a stochastic process that counts the number of events and the time points at which these events occur in a given time interval. The time between each pair of consecutive events has an exponential distribution with parameter λ and each of these inter-arrival times is assumed to be independent of other inter-arrival times.

An infection rate is an estimate of the rate of progress of a disease, based on proportional measures of the extent of infection at different times.

In probability theory, the probability generating function of a discrete random variable is a power series representation of the probability mass function of the random variable. Probability generating functions are often employed for their succinct description of the sequence of probabilities $\Pr(X= i)$ in the probability mass function for a random variable X , and to make available the well developed theory of power series with non-negative coefficients.

Definition 1.3 ^[10]

Consider a random variable X , i.e, a discrete random variable taking non-negative values.

Write $p_k = P(X = k)$, $k = 0, 1, 2, \dots$

The probability generating function of X is defined as

$$G_X(S) = \sum_{k=0}^{\infty} P_k S^k = E(S^X).$$

Note that $G_X(1) = 1$, so the series converges absolutely for

Theorem 1.4 ^[10] (Total and Compound Probability)

Let A_1, A_2, \dots, A_n be a partition of Ω . For any event B ,

$$\Pr(B) = \sum_{j=1}^n \Pr(A_j) \Pr(B/A_j).$$

In this paper, we have derived two stochastic epidemic models one is with Poisson infection and carrier rates another with Poisson infection, carrier and removal rates.

2. Stochastic Epidemic Model with Carriers, Infectives and Carriers

In this section, we consider a disease which spread only by carriers so that our interest is in the two classes of individuals, namely, susceptibles and carriers. Carrier a person or animal that harbours the infectious agent for a disease and can transmit it to others, but does not demonstrate signs of the disease. A carrier can be asymptomatic (never indicate signs of the disease) or can display signs of the disease only during the incubation period, convalescence, or post convalescence. The period of being a carrier can be short (a transient carrier) or long (a chronic carrier). Carriers are eliminated by external action. Thus we get the following model.

2.1. Stochastic Epidemic Model with Carriers with Poisson Infection and Poisson Carrier Rates

Let $P_{m,n}(t)$ be the probability that there are m susceptibles and n carriers in the population at time t . If N is the total size of the population, then the number of persons in the removed category is $N-m-n$.

Let the probability of susceptible being infected in the time interval $(t, t + \Delta t)$ be $\beta mn \Delta t + O(\Delta t)$, and let the corresponding probability of one carrier being removed in the same time interval be $\gamma n \Delta t + O(\Delta t)$.

In this case, let us assume that βmn is the Poisson rate of susceptible becomes infected with parameter λ_I , and γn is the Poisson rate of carrier with parameter λ_C .

i.e.,
$$\beta mn = \frac{e^{-\lambda_I} (\lambda_I)^m}{m!}, \quad m \geq 0$$
 (1)

and
$$\gamma n = \frac{e^{-\lambda_C} (\lambda_C)^n}{n!}, \quad n \geq 0.$$
 (2)

Here, λ_I carrier be the rate of infection from susceptible to infected, and λ_C be the rate of carrier.

The probability that there is no change in the time interval $(t, t + \Delta t)$ is then given by

$$1 - \beta mn \Delta t - \gamma n \Delta t + O(\Delta t). \tag{3}$$

Now there can be m susceptibles and n carriers at time $t + \Delta t$ if there are

- $m+1$ susceptibles and n carriers at time t and if one person has become infected in time Δt , (or)
- m susceptibles and $n+1$ carriers at time t and if one carrier has been removed in time Δt , (or)
- m susceptibles and n carriers at time t and if there is no change in time Δt .

We assume that the probability of more than one change in time Δt is $O(\Delta t)$.

Then, using the theorems of total and compound probability, we get

$$P_{m,n}(t + \Delta t) = P_{m+1,n}(t) \beta (m+1) n \Delta t + P_{m,n+1}(t) \gamma (n+1) \Delta t + P_{m,n}(t) (1 - \beta mn \Delta t - \gamma n \Delta t) + O(\Delta t) \tag{4}$$

So that,

$$P_{m,n}(t + \Delta t) = P_{m+1,n}(t) \beta (m+1) n \Delta t + P_{m,n+1}(t) \gamma (n+1) \Delta t + P_{m,n}(t) - P_{m,n}(t) \beta mn \Delta t - P_{m,n}(t) \gamma n \Delta t + O(\Delta t)$$

$$P_{m,n}(t + \Delta t) - P_{m,n}(t) = P_{m+1,n}(t) \beta (m+1) n \Delta t + P_{m,n+1}(t) \gamma (n+1) \Delta t - P_{m,n}(t) \beta mn \Delta t - P_{m,n}(t) \gamma n \Delta t + O(\Delta t)$$

Dividing on both sides by Δt , we get

$$\begin{aligned} \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= P_{m+1,n}(t) \beta (m+1) n + P_{m,n+1}(t) \gamma (n+1) \\ &\quad - P_{m,n}(t) \beta mn - P_{m,n}(t) \gamma n + \frac{O(\Delta t)}{\Delta t} \end{aligned} \tag{5}$$

Proceeding to the limit as $\Delta t \rightarrow 0$ in (5), we get

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{P_{m,n}(t + \Delta t) - P_{m,n}(t)}{\Delta t} &= P_{m+1,n}(t) \beta (m+1) n - P_{m,n}(t) \beta mn \\ &\quad + P_{m,n+1}(t) \gamma (n+1) - P_{m,n}(t) \gamma n + 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dt} (P_{m,n}(t)) &= \beta (m+1) n P_{m+1,n}(t) - \beta mn P_{m,n}(t) \\ &\quad + \gamma (n+1) P_{m,n+1}(t) - \gamma n P_{m,n}(t) \end{aligned} \tag{6}$$

Initially, let there be a susceptibles and b carriers. Then we define the probability generating function by

$$\varphi(x, y, t) = \sum_{m=0}^a \sum_{n=0}^{a+b-m} P_{m,n}(t) x^m y^n \tag{7}$$

Multiplying (6) by $x^m y^n$ and summing over n from 0 to $a+b-m$ and m from 0 to a , we get

$$\frac{\partial}{\partial t} \left(\sum_{m=0}^a \sum_{n=0}^{a+b-m} P_{m,n}(t) x^m y^n \right) = \sum_{m=0}^a \sum_{n=0}^{a+b-m} \frac{d}{dt} (P_{m,n}(t)) x^m y^n$$

So that,

$$\frac{\partial \varphi}{\partial t} = \sum_{m=0}^a \sum_{n=0}^{a+b-m} \left[\beta(m+1)n P_{m+1,n}(t) - \beta mn P_{m,n}(t) + \gamma(n+1) P_{m,n+1}(t) - \gamma n P_{m,n}(t) \right] x^m y^n$$

(Using (6) and (7))

$$= \sum_{m=0}^a \sum_{n=0}^{a+b-m} (\beta(m+1)n P_{m+1,n}(t)) x^m y^n - \sum_{m=0}^a \sum_{n=0}^{a+b-m} (\beta mn P_{m,n}(t)) x^m y^n$$

$$+ \sum_{m=0}^a \sum_{n=0}^{a+b-m} (\gamma(n+1) P_{m,n+1}(t)) x^m y^n - \sum_{m=0}^a \sum_{n=0}^{a+b-m} (\gamma n P_{m,n}(t)) x^m y^n$$

$$\frac{\partial \varphi}{\partial t} = \beta \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} ((m+1)n P_{m+1,n}(t)) x^m y^n - \sum_{m=0}^a \sum_{n=0}^{a+b-m} mn P_{m,n}(t) x^m y^n \right\}$$

$$+ \gamma \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} ((n+1) P_{m,n+1}(t)) x^m y^n - \sum_{m=0}^a \sum_{n=0}^{a+b-m} n P_{m,n}(t) x^m y^n \right\}$$

(8)

Now, the definition of probability generating function, we have

$$x \frac{\partial \varphi}{\partial x} = \sum_m \sum_n m P_{m,n}(t) x^m y^n ;$$

(9)

$$y \frac{\partial \varphi}{\partial y} = \sum_m \sum_n n P_{m,n}(t) x^m y^n ;$$

(10)

and

$$xy \frac{\partial^2 \varphi}{\partial x \partial y} = \sum_m \sum_n mn P_{m,n}(t) x^m y^n$$

(11)

Equation (8) can be written as,

$$\frac{\partial \varphi}{\partial t} = \beta \left\{ y \sum_{m=0}^a \sum_{n=0}^{a+b-m} ((m+1)n P_{m+1,n}(t)) x^m y^{n-1} - xy \sum_{m=0}^a \sum_{n=0}^{a+b-m} mn P_{m,n}(t) x^{m-1} y^{n-1} \right\}$$

$$+ \gamma \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} ((n+1) P_{m,n+1}(t)) x^m y^n - y \sum_{m=0}^a \sum_{n=0}^{a+b-m} n P_{m,n}(t) x^m y^{n-1} \right\}$$

(12)

By using the relations (9), (10) and (11) in (12), we get

$$\frac{\partial \varphi}{\partial t} = \beta \left\{ y \frac{\partial^2 \varphi}{\partial x \partial y} - xy \frac{\partial^2 \varphi}{\partial x \partial y} \right\} + \gamma \left\{ \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial y} \right\}$$

$$\frac{\partial \varphi}{\partial t} = \beta (y - xy) \frac{\partial^2 \varphi}{\partial x \partial y} + \gamma (1 - y) \frac{\partial \varphi}{\partial y}$$

(13)

In (13), we allow β and γ to be functions of t , so that we get

$$\frac{\partial \varphi}{\partial t} = \beta(t)(y - xy) \frac{\partial^2 \varphi}{\partial x \partial y} + \gamma(t)(1 - y) \frac{\partial \varphi}{\partial y}$$

$$\text{i.e., } \frac{\partial \varphi}{\partial t} = \beta(t)(1 - x) y \frac{\partial^2 \varphi}{\partial x \partial y} + \gamma(t)(1 - y) \frac{\partial \varphi}{\partial y} \tag{14}$$

We have to solve this subject to the initial condition,

$$\varphi(x, y, 0) = x^M y^N \tag{15}$$

where M, N are the initial number of susceptibles and carriers.

We try the solution

$$\varphi(x, y, t) = \sum_{r=0}^M {}^M C_r (x-1)^r f_r(y, t) \tag{16}$$

Now,

$$\frac{\partial \varphi}{\partial t} = \sum_{r=0}^M {}^M C_r (x-1)^r \frac{\partial f_r}{\partial t}; \tag{17}$$

$$\frac{\partial \varphi}{\partial y} = \sum_{r=0}^M {}^M C_r (x-1)^r \frac{\partial f_r}{\partial y}; \text{ and} \tag{18}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial y} \right) = \sum_{r=0}^M {}^M C_r r (x-1)^{r-1} \frac{\partial f_r}{\partial y} \tag{19}$$

By using (17), (18) and (19) in (14), we get

$$\sum_{r=0}^M {}^M C_r (x-1)^r \frac{\partial f_r}{\partial t} = \beta(1-x) y \sum_{r=0}^M {}^M C_r r (x-1)^{r-1} \frac{\partial f_r}{\partial y} + \gamma(1-y) \sum_{r=0}^M {}^M C_r (x-1)^r \frac{\partial f_r}{\partial y}$$

$$\sum_{r=0}^M {}^M C_r (x-1)^r \frac{\partial f_r}{\partial t} + \sum_{r=0}^M {}^M C_r (x-1)^r r \beta y \frac{\partial f_r}{\partial y} - \sum_{r=0}^M {}^M C_r (x-1)^r \gamma (1-y) \frac{\partial f_r}{\partial y} = 0$$

$$\sum_{r=0}^M {}^M C_r (x-1)^r \left\{ \frac{\partial f_r}{\partial t} + r \beta y \frac{\partial f_r}{\partial y} - \gamma (1-y) \frac{\partial f_r}{\partial y} \right\} = 0$$

$$\sum_{r=0}^M {}^M C_r (x-1)^r \left\{ \frac{\partial f_r}{\partial t} + (r \beta y - \gamma + \gamma y) \frac{\partial f_r}{\partial y} \right\} = 0$$

$$\text{i.e., } \sum_{r=0}^M {}^M C_r (x-1)^r \left\{ \frac{\partial f_r}{\partial t} + ((r \beta + \gamma) y - \gamma) \frac{\partial f_r}{\partial y} \right\} = 0 \tag{20}$$

Since the polynomials

$$P_r(x) = {}^M C_r (x-1)^r, \quad r = 0, 1, 2, \dots, M \tag{21}$$

are linearly independent, we get

$$\frac{\partial f_r}{\partial t} + ((r \beta + \gamma) y - \gamma) \frac{\partial f_r}{\partial y} = 0 \tag{22}$$

For each r , this is a linear partial differential equation of the first order. The auxiliary equations for solving (22) are

$$\frac{dt}{1} = \frac{dy}{(r\beta + \gamma)y - \gamma} = \frac{df_r}{0} \tag{23}$$

An intermediate integral is obtained by solving

$$\frac{dy}{dt} = (r\beta + \gamma)y - \gamma$$

i.e.,
$$\frac{dy}{dt} - (r\beta + \gamma)y = -\gamma \tag{24}$$

So that if
$$F_r(t) = \int_0^t (r\beta(t) + \gamma(t)) dt \tag{25}$$

Now, the solution is

$$f_r(y, t) = \psi_r \left\{ y \exp(-F_r(t)) + \int_0^t \gamma(t) \exp(-F_r(t)) dt \right\} \tag{26}$$

But from (15) and (16),

$$\varphi(x, y, 0) = \sum_{r=0}^M {}^M C_r (x-1)^r f_r(y, 0) = x^M y^N \tag{27}$$

which is satisfied if

$$f_r(y, 0) = y^N \tag{28}$$

So that

$$f_r(y, t) = \left\{ y \exp(-F_r(t)) + \int_0^t \gamma(t) \exp(-F_r(t)) dt \right\}^N \tag{29}$$

By using (29) in (16), we get

$$\varphi(x, y, t) = \sum_{r=0}^M {}^M C_r (x-1)^r \left\{ y \exp(-F_r(t)) + \int_0^t \gamma(t) \exp(-F_r(t)) dt \right\}^N \tag{30}$$

From this we deduce

$$\begin{aligned} E(m(t)) &= \left(\frac{\partial \varphi}{\partial x} \right)_{x=1, y=1} \\ &= {}^M C_1 \left[\exp(-F_1(t)) + \int_0^t \gamma(t) \exp(-F_1(t)) dt \right]^N \end{aligned} \tag{31}$$

$$\begin{aligned}
 E(n(t)) &= \left(\frac{\partial \varphi}{\partial y} \right)_{x=1, y=1} \\
 &= N \left[\exp(-F_0(t)) + \int_0^t \gamma(t) \exp(-F_0(t)) dt \right]^{N-1} \exp(-F_0(t))
 \end{aligned}
 \tag{32}$$

Similarly, we can find variances and covariances of the number of susceptible and carriers.

2.2. Stochastic Epidemic Model with Infectives and Carriers

Let m, n, p denote the number of susceptibles, infectives and carriers. A susceptible can become infective by contact with either an infected or a carrier. The result is represented in the following model.

Let $P_{m,n,p}(t)$ be the probability that there are m susceptibles, n infectives and p carriers in the population at time t . If N is the total size of the population, then the number of persons in the removed category is $N-m-n-p$.

Let the probability of a susceptible being infected in the time interval $(t, t + \Delta t)$ be $\beta mn \Delta t + O(\Delta t)$, and let the probability of a susceptible being infected due to contact with a carrier in the time interval $(t, t + \Delta t)$ be $\gamma mp \Delta t + O(\Delta t)$, and also let the corresponding probability of one carrier being removed in the same time interval be $\delta p \Delta t + O(\Delta t)$.

In this case, let us assume that βmn is the Poisson rate of susceptible becomes infected with parameter λ_I , γmp is the Poisson rate of susceptible becomes infected due to contact with a carrier with parameter λ_C , and δp is the Poisson rate of carrier becomes removed with parameter λ_R .

$$\text{i.e., } \beta mn = \frac{e^{-\lambda_I} (\lambda_I)^m}{m!}, \quad m \geq 0
 \tag{33}$$

$$\gamma mp = \frac{e^{-\lambda_C} (\lambda_C)^m}{m!}, \quad m \geq 0
 \tag{34}$$

$$\text{and } \delta p = \frac{e^{-\lambda_R} (\lambda_R)^p}{p!}, \quad p \geq 0.
 \tag{35}$$

Here, λ_I be the rate of infection from susceptible to infected, λ_C be the rate of carrier and λ_R be the rate of removal from carrier to removed.

The probability that there is no change in the time interval $(t, t + \Delta t)$ is then given by

$$1 - \beta mn \Delta t - \delta p \Delta t + O(\Delta t)
 \tag{36}$$

Now there can be m susceptibles, n infectives and p carriers at time $t + \Delta t$ if there are

- a) $m+1$ susceptibles, $n-1$ infectives and p carriers at time t and if one person has become infected in time Δt , (or)
- b) $m+1$ susceptibles, $n-1$ infectives and p carriers at time t and if one person has become infective due to contact with a carrier in time Δt , (or)
- c) m susceptibles, n infectives and $p+1$ carriers at time t and if one carrier has been removed in time Δt , (or)
- d) m susceptibles, n infectives and p carriers at time t and if there is no change in time Δt .

We assume that the probability of more than one change in time Δt is $O(\Delta t)$.

Then, using the theorems of total and compound probability, we get

$$\begin{aligned}
 P_{m,n,p}(t + \Delta t) &= P_{m+1,n-1,p}(t) \beta (m+1)(n-1) \Delta t + P_{m+1,n-1,p}(t) \gamma (m+1) p \Delta t \\
 &\quad + P_{m,n,p+1}(t) \delta (p+1) \Delta t + P_{m,n,p}(t) (1 - \beta mn \Delta t - \delta p \Delta t) + O(\Delta t)
 \end{aligned}
 \tag{37}$$

So that,

$$\begin{aligned}
 P_{m,n,p}(t + \Delta t) &= P_{m+1,n-1,p}(t) \beta(m+1)(n-1)\Delta t + P_{m+1,n-1,p}(t) \gamma(m+1) p \Delta t \\
 &\quad + P_{m,n,p+1}(t) \delta(p+1)\Delta t + P_{m,n,p}(t) - P_{m,n,p}(t) \beta mn \Delta t \\
 &\quad - P_{m,n,p}(t) \delta p \Delta t + O(\Delta t) \\
 P_{m,n,p}(t + \Delta t) - P_{m,n,p}(t) &= P_{m+1,n-1,p}(t) \beta(m+1)(n-1)\Delta t + P_{m+1,n-1,p}(t) \gamma(m+1) p \Delta t \\
 &\quad + P_{m,n,p+1}(t) \delta(p+1)\Delta t - P_{m,n,p}(t) \beta mn \Delta t \\
 &\quad - P_{m,n,p}(t) \delta p \Delta t + O(\Delta t)
 \end{aligned} \tag{38}$$

Dividing on both sides by Δt , and proceeding to the limit as $\Delta t \rightarrow 0$ in (38), we get

$$\begin{aligned}
 \lim_{\Delta t \rightarrow 0} \frac{P_{m,n,p}(t + \Delta t) - P_{m,n,p}(t)}{\Delta t} &= P_{m+1,n-1,p}(t) \beta(m+1)(n-1) + P_{m+1,n-1,p}(t) \gamma(m+1) p \\
 &\quad + P_{m,n,p+1}(t) \delta(p+1) - P_{m,n,p}(t) \beta mn \\
 &\quad - P_{m,n,p}(t) \delta p + 0
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d}{dt}(P_{m,n,p}(t)) &= \beta(m+1)(n-1) P_{m+1,n-1,p}(t) - \beta mn P_{m,n,p}(t) + \gamma(m+1) p P_{m+1,n-1,p}(t) \\
 &\quad + \delta(p+1) P_{m,n,p+1}(t) - \delta p P_{m,n,p}(t)
 \end{aligned} \tag{39}$$

Initially, let there be a susceptibles, b infectives and c carriers. Then we define the probability generating function by

$$\frac{\partial}{\partial t} \left(\sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} P_{m,n,p}(t) x^m y^n z^p \right) = \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} \frac{d}{dt} (P_{m,n,p}(t)) x^m y^n z^p \tag{40}$$

Multiplying (39) by $x^m y^n z^p$ and summing over p from 0 to $a+b+c-m-n$; n from 0 to $a+b-m$ and m from 0 to a , we get

$$\frac{\partial}{\partial t} \left(\sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} P_{m,n,p}(t) x^m y^n z^p \right) = \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} \frac{d}{dt} (P_{m,n,p}(t)) x^m y^n z^p$$

So that,

$$\begin{aligned}
 \frac{\partial \phi}{\partial t} &= \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} \left\{ \beta(m+1)(n-1) P_{m+1,n-1,p}(t) - \beta mn P_{m,n,p}(t) \right. \\
 &\quad \left. + \gamma(m+1) p P_{m+1,n-1,p}(t) + \delta(p+1) P_{m,n,p+1}(t) - \delta p P_{m,n,p}(t) \right\} x^m y^n z^p \\
 &\hspace{15em} \text{(Using (39) and (40))} \\
 &= \beta \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (m+1)(n-1) P_{m+1,n-1,p}(t) x^m y^n z^p - \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} mn P_{m,n,p}(t) x^m y^n z^p \right\} \\
 &\quad + \gamma \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (m+1) p P_{m+1,n-1,p}(t) x^m y^n z^p \right\} \\
 &\quad + \delta \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (p+1) P_{m,n,p+1}(t) x^m y^n z^p - \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} p P_{m,n,p}(t) x^m y^n z^p \right\} \tag{41}
 \end{aligned}$$

Now, the definition of probability generating function, we have

$$x \frac{\partial \phi}{\partial x} = \sum_m \sum_n \sum_p m P_{m,n,p}(t) x^m y^n z^p; \tag{42}$$

$$y \frac{\partial \phi}{\partial y} = \sum_m \sum_n \sum_p n P_{m,n,p}(t) x^m y^n z^p; \tag{43}$$

$$z \frac{\partial \phi}{\partial z} = \sum_m \sum_n \sum_p p P_{m,n,p}(t) x^m y^n z^p; \tag{44}$$

$$xy \frac{\partial^2 \phi}{\partial x \partial y} = \sum_m \sum_n \sum_p mn P_{m,n,p}(t) x^m y^n z^p; \tag{45}$$

and

$$xz \frac{\partial^2 \phi}{\partial x \partial z} = \sum_m \sum_n \sum_p mp P_{m,n,p}(t) x^m y^n z^p. \tag{46}$$

Equation (41) can be written as,

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & \beta \left\{ y^2 \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (m+1)(n-1) P_{m+1,n-1,p}(t) x^m y^{n-2} z^p \right. \\ & \left. - xy \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} mn P_{m,n,p}(t) x^{m-1} y^{n-1} z^p \right\} \\ & + \gamma \left\{ z \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (m+1) p P_{m+1,n-1,p}(t) x^m y^n z^{p-1} \right\} \\ & + \delta \left\{ \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} (p+1) P_{m,n,p+1}(t) x^m y^n z^p \right. \\ & \left. - z \sum_{m=0}^a \sum_{n=0}^{a+b-m} \sum_{p=0}^{a+b+c-m-n} p P_{m,n,p}(t) x^m y^n z^{p-1} \right\} \tag{47} \end{aligned}$$

By using the relations (9), (10), (11), (12) and (13) in (14), we get

$$\frac{\partial \phi}{\partial t} = \beta \left(y^2 \frac{\partial^2 \phi}{\partial x \partial y} - xy \frac{\partial^2 \phi}{\partial x \partial y} \right) + \gamma \left(z \frac{\partial^2 \phi}{\partial x \partial z} \right) + \delta \left(\frac{\partial \phi}{\partial z} - z \frac{\partial \phi}{\partial z} \right)$$

$$\frac{\partial \phi}{\partial t} = \beta (y^2 - xy) \frac{\partial^2 \phi}{\partial x \partial y} + \gamma z \frac{\partial^2 \phi}{\partial x \partial z} + \delta (1 - z) \frac{\partial \phi}{\partial z}$$

that is,

$$\tag{48}$$

2.3. Classification of the solutions

In this section, to find the classification of the solution of the partial differential equation (48) derived in section 2.2.

Now, the given equation (48) is a linear partial differential equation of the second order in four independent variables x,y,z and t .

By using the result of section A.4.7 in [11], equation (48) can be re-written as

$$\begin{aligned}
 &0 \cdot \varphi_{xx} + \frac{\beta}{2}(y^2 - xy)\varphi_{xy} + \frac{\gamma}{2}z\varphi_{xz} + 0 \cdot \varphi_{xt} + \frac{\beta}{2}(y^2 - xy)\varphi_{yx} \\
 &+ 0 \cdot \varphi_{yy} + 0 \cdot \varphi_{yz} + 0 \cdot \varphi_{yt} + \frac{\gamma}{2}z\varphi_{yx} + 0 \cdot \varphi_{zy} + 0 \cdot \varphi_{zz} + 0 \cdot \varphi_{zt} + 0 \cdot \varphi_{tx} \\
 &+ 0 \cdot \varphi_{ty} + 0 \cdot \varphi_{tz} + 0 \cdot \varphi_{tt} + 0 \cdot \varphi_x + 0 \cdot \varphi_y + \delta(1-z)\varphi_z - 1 \cdot \varphi_t + 0 \cdot \varphi = 0
 \end{aligned} \tag{49}$$

Let us define the matrix A , is given by

$$A = \begin{bmatrix} \text{coeff. of } u_{xx} & \text{coeff. of } u_{xy} & \text{coeff. of } u_{xz} & \text{coeff. of } u_{xt} \\ \text{coeff. of } u_{yx} & \text{coeff. of } u_{yy} & \text{coeff. of } u_{yz} & \text{coeff. of } u_{yt} \\ \text{coeff. of } u_{zx} & \text{coeff. of } u_{zy} & \text{coeff. of } u_{zz} & \text{coeff. of } u_{zt} \\ \text{coeff. of } u_{tx} & \text{coeff. of } u_{ty} & \text{coeff. of } u_{tz} & \text{coeff. of } u_{tt} \end{bmatrix} \tag{50}$$

Therefore,

$$A = \begin{bmatrix} 0 & \frac{\beta}{2}(y^2 - xy) & \frac{\gamma}{2}z & 0 \\ \frac{\beta}{2}(y^2 - xy) & 0 & 0 & 0 \\ \frac{\gamma}{2}z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{Using (49) and (50)}$$

Since, $a_{ij} = a_{ji}$, $A = [a_{ij}]_{4 \times 4}$ is a symmetric matrix of order 4×4 .

So that,

$$|A| = \begin{vmatrix} 0 & \frac{\beta}{2}(y^2 - xy) & \frac{\gamma}{2}z & 0 \\ \frac{\beta}{2}(y^2 - xy) & 0 & 0 & 0 \\ \frac{\gamma}{2}z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \tag{51}$$

Also, the Eigen values of A is given by the form $|A - \lambda I| = 0$

That is,
$$\begin{vmatrix} -\lambda & \frac{\beta}{2}(y^2 - xy) & \frac{\gamma}{2}z & 0 \\ \frac{\beta}{2}(y^2 - xy) & -\lambda & 0 & 0 \\ \frac{\gamma}{2}z & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda^4 - \lambda^2 \frac{\beta^2}{4}(y^2 - xy)^2 - \lambda^2 \frac{\gamma^2}{4}z^2 = 0$$

$$\lambda^2 \left(\lambda^2 - \frac{\beta^2}{4}(y^2 - xy)^2 - \frac{\gamma^2}{4}z^2 \right) = 0$$

$$\lambda = 0, 0; \quad \lambda^2 = \frac{\beta^2}{4}(y^2 - xy)^2 + \frac{\gamma^2}{4}z^2$$

$$\lambda = 0, 0; \quad \lambda = \pm \frac{1}{2} \sqrt{\beta^2 (y^2 - xy)^2 + \gamma^2 z^2}.$$

Therefore,

(52)

In equations (51) and (52), the determinant value of A is zero and one of the Eigen values of A is also zero.

By using case (ii) of section A.4.7 in [11], the solution of the given partial differential equation (48) is of parabolic type.

Conclusion

Here we have developed two stochastic epidemic models by considering infectives, carriers and removal rates. Hence there are three types of removal recovery, immunization and death this can be elaborately studied by considering each of the removals separately.

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