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On the binary quadratic Diophantine equation $x^2 - 5xy + y^2 + 30x = 0$

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Abstract
The binary quadratic equation $x^2 - 5xy + y^2 + 30x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

Keywords: Binary quadratic equation, integral solutions

1. Introduction
The binary quadratic Diophantine equations (both homogeneous and non-homogeneous) are rich in variety [1-6]. In [7-16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of another interesting binary quadratic equation given by $x^2 - 5xy + y^2 + 30x = 0$.
The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

Method of Analysis
The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 5xy + y^2 + 30x = 0 \tag{1}$$

Note that (1) is satisfied by the following non-zero integer pairs
$(-2,4)(4,-8)(6,12)(6,18)(10,10)(12,24)$

However, we have other solutions for (1), which are illustrated below:
Solving (1) for y, we have

$$y = \frac{1}{2}\left[5x \pm \sqrt{25x^2 - 4\left(x^2 + 30x\right)}\right]$$

$$y = \frac{1}{2}\left[5x \pm \sqrt{21x^2 - 120x}\right] \tag{2}$$

Let $\alpha^2 = 21x^2 - 120x$

Multiplying the above equation by 21 on both sides and performing a few calculations, we have

$$X^2 = 21\alpha^2 + 60^2 \tag{3}$$

where

$$X = 21x - 60 \tag{4}$$

The least positive integer solution of (3) is

$$\alpha_0 = 6, X_0 = 66$$

Now to find the other solutions of (3), consider the pellian equation

$$X^2 = 21\alpha^2 + 1 \quad (5)$$

whose fundamental solution is

$$(\tilde{\alpha}_0, \tilde{X}_0) = (12,55)$$

The other solutions of (5) can be derived from the relations

$$\tilde{X}_n = \frac{f_n}{2}, \tilde{\alpha}_n = \frac{g_n}{2\sqrt{21}}$$

where
$$f_n = (55+12\sqrt{21})^{n+1} + (55-12\sqrt{21})^{n+1}$$
$$g_n = (55+12\sqrt{21})^{n+1} - (55-12\sqrt{21})^{n+1}$$

Applying the lemma of Brahmagupta between (α_0, X_n) and $(\tilde{\alpha}_n, \tilde{X}_n)$, the other solutions of (3) can be obtained from the relations.

$$\alpha_{n+1} = 6\tilde{X}_n + 66\tilde{\alpha}_n \quad (6)$$

$$X_{n+1} = 66\tilde{X}_n + 126\tilde{\alpha}_n \quad (7)$$

Taking positive sign on R.H.S of (2) and using (4), (6) and (7), we have

$$x_{n+1} = \frac{1}{21}\left[60 + 33 f_n + 3\sqrt{21} g_n\right] \quad (8)$$

$$y_{n+1} = \frac{1}{21}\left[150 + 114 f_n + 24\sqrt{21} g_n\right], n = -1,1,3..... \quad (9)$$

which represents the integer solutions to (1)

The recurrence relations for x_{n+1}, y_{n+1} are respectively

$$x_{n+5} - 12098 x_{n+3} + x_{n+1} = -34560$$

$$y_{n+5} - 12098 y_{n+3} + y_{n+1} = -86400$$

A few numerical examples are given in table below

Table: Numerical Solutions

n	x_{n+1}	y_{n+1}
-1	6	18
1	26934	129042
3	325812966	1561063698
5	3941685201000	18885748980000

Some relations satisfied by the solutions (8) and (9) are as follows

1) $x_{n+3} = 2640 y_{n+1} - 551 x_{n+1} - 17280$
2) $x_{n+5} = 31938720 y_{n+1} - 6665999 x_{n+1} - 209088000$
3) $y_{n+3} = 12649 y_{n+1} - 2640 x_{n+1} - 82800$
4) $y_{n+5} = 153027601 y_{n+1} - 31938720 x_{n+1} - 1001800800$

Each of the following expression is a nasty number

i. $168 x_{n+1} - 21 y_{n+1} - 30$
ii. $168 x_{n+1} - 21 y_{n+1} - 630$

Remarkable Observations

1) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for other choices of hyperbola

For example, define

$$X = 168x_{n+1} - 21y_{n+1} - 330$$

$$Y = 231y_{n+1} - 798x_{n+1} + 630$$

Note that the pairs (X, Y) satisfies the hyperbola

$$Y^2 = 21X^2 - 84(150)^2$$

2) By considering suitable linear transformations between the solutions of (1), one may get integer solutions for other choices of parabola

As an example, consider

$$X = 168x_{2n+2} - 21y_{2n+2} - 30$$

$$Y = 231y_{n+1} - 798x_{n+1} + 630$$

Note that the pair (X, Y) satisfied the parabola

$$Y^2 = (150 \times 21)X - 84(150)^2$$

Note

Treating (1) as a quadratic in x and solving for x, we get

$$x = \frac{1}{2}\left[(5y - 30) + \sqrt{21y^2 - 300y + 900}\right]$$

Following the procedure as above, the corresponding values of x and y satisfying (1) are given by

$$x_{n+1} = \frac{1}{21}\left[60 + 114f_n + 24\sqrt{21}g_n\right]$$

$$y_{n+1} = \frac{1}{21}\left[150 + 33f_n + 3\sqrt{21}g_n\right]$$

The corresponding recurrence relations satisfied by x_{n+1} and y_{n+1} are as follows:

$$x_{n+5} - 12098x_{n+3} + x_{n+1} = -34560$$

$$y_{n+5} - 12098y_{n+3} + y_{n+1} = -86400$$

A few interesting properties observed between the solutions are given below:

$$x_{n+3} = 12649x_{n+1} - 2640y_{n+1} - 17280$$

$$x_{n+5} = 153027601x_{n+1} - 31938720y_{n+1} - 209087999$$

$$y_{n+3} = 2640x_{n+1} - 551y_{n+1} - 3600$$

$$y_{n+5} = 31938720x_{n+1} - 6665999y_{n+1} - 43639200$$

Conclusion

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

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