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On the negative pell equation $y^2 = 105\,x^2 - 5$

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Abstract

The negative pell equation represented by the binary quadratic equation $y^2 = 105x^2 - 5$ is analysed for its non-zero distinct integer solutions. A few interesting relations among the solutions are presented. Employing the solutions of the equation under consideration, the integer solutions for a few choices of hyperbola and parabola are obtained.

Keywords: Binary quadratic, Hyperbola, parabola, Integral solutions, pell equation. 2010 Mathematics subject classification: 11D09

1. Introduction

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer, is known as pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over the world, since antiquity, J.L.lagrange proved that the positive pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solutions, where as $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-8]. More specifically, one may refer " **The On-line Encyclopedia of integer sequences" (A031396, A130226, A031398)** for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative pell equation given by $y^2 = 105x^2 - 5$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2. Method of analysis

The negative pell equation represented hyperbola under consideration is

$$y^2 = 105x^2 - 5 \qquad (1)$$

Whose smallest positive integer solution $x_0 = 1, y_0 = 10$

To obtain the other solutions of (1), consider the pell equation

$$y^2 = 105x^2 + 1 \qquad (2)$$

Whose general solution is given by

$$\widetilde{x_n} = \frac{g_n}{2\sqrt{105}} \quad ; \quad \widetilde{y_n} = \frac{f_n}{2}$$

Where $f_n = \left(41 + 4\sqrt{105}\right)^{n+1} + \left(41 - 4\sqrt{105}\right)^{n+1}$

$$g_n = \left(41 + 4\sqrt{105}\right)^{n+1} - \left(41 - 4\sqrt{105}\right)^{n+1} \quad , n = 0, 1, 2, 3\ldots..$$

Applying Brahma Gupta lemma between (x_0, y_0) and ($\widetilde{x_n}, \widetilde{y_n}$) the other in (1) are given by

$$x_{n+1} = \frac{f_n}{2} + \frac{\sqrt{105}}{21} g_n \quad ; \quad y_{n+1} = 5f_n + \frac{\sqrt{105}}{21} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 82x_{n+2} + x_{n+1} = 0, x_0 = 1, x_1 = 81$$

$$y_{n+3} - 82y_{n+2} + y_{n+1} = 0, y_0 = 10, y_1 = 830$$

Some numerical examples of x and y satisfying (1) are given in the following ta

n	x_n	y_n
0	1	10
1	81	830
2	6641	68050
3	544481	5579270
4	44640801	457432090
5	3660001201	37503852110
6	300075457681	3074858440930
7	24602527528641	252100888304150

From the above table, we observe some interesting relations among the solutions which are presented below

1. x_n is always odd
2. y_n is always even
3. $x_{n+1} \equiv 0 (mod\ 3)$
4. $y_{n+1} \equiv 0(\text{mod } 10)$
5. $6(42x_{2n+2} - 4y_{2n+2} + 2)$ is a nasty number
6. $42_{3n+3} - 4y_{3n+3} + 3[42x_{n+1} - 4y_{n+1}]$ is a cubical number
7. $840t_{3,n+} + 100$ is a perfect square, where $t_{3,n}$ is the n^{th} triangular number
8. $x_{n+2} = 41x_{n+1} + 4y_{n+1}$
9. $x_{n+3} = 3361x_{n+1} + 328y_{n+1}$
10. $y_{n+2} = 420x_{n+1} + 41y_{n+1}$
11. $y_{n+3} = 34440x_{n+1} + 3361y_{n+1}$
12. $x_{n+1}y_{n+3} - x_{n+3}y_{n+1} = 1640$
13. $105x_{n+1}x_{n+2} - y_{n+1}y_{n+2} = 205$
14. $(82x_{n+2} - y_{n+3})y_{n+2} - x_{n+2}(82x_{n+2} - y_{n+3}) = 20$
15. $105x_{n+3}x_{n+1} - y_{n+1}y_{n+3} = 16805$
16. $16805(420x_{n+1} + 41y_{n+1}) = (2100x_{n+3}) + (82y_{n+2} - y_{n+3})$
$* (105x_{n+2}x_{n+3} - y_{n+2}y_{n+3})$
17. $105(42x_{3n+3} - 4y_{3n+3} + 126x_{n+1} - 12y_{n+1}) - (42y_{n+1} - 420x_{n+1})^2 *$
$(42x_{n+1} - 4y_{n+1}) = 42x_{n+1} - 4y_{n+1}$

3. Remarkable observations

1. Define $X = (4y_{3n+3} - 42x_{3n+3}) + (42x_{n+1} - 4y_{n+1})^3$

$Y = 42y_{n+1} - 420x_{n+1}$

Note that the pair (X, Y) satisfies the hyperbola $105X^2 - 9Y^2 = 3780$

2. Define $\alpha_1 = 42x_{2n+2} - 4y_{2n+2}$

$\beta_1 = 42y_{n+1} - 420x_{n+1}$

Observe that the pair (α_1, β_1) satisfies the hyperbola $\beta_1^2 = 105\alpha_1^2 - 420$

3. Define $\alpha_2 = (42x_{n+1} - 4y_{n+1})^3 + 4y_{3n+3} - 42x_{3n+3}$

$\beta_2 = y_{n+1} - 10x_{n+1}$

4. Define $\alpha_3 = 42x_{2n+2} - 4y_{2n+2}$

$\beta_3 = 42y_{n+1} - 420x_{n+1}$

Note that the pair (α_3, β_3) satisfies the parabola $\beta_3^2 = 105\alpha_3 - 420$

4. Conclusion

In this paper , We have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 105x^2 - 5$, As the binary quadratic Diophantine equation are rich in variety , one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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