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S. Vidhyalakshmi
Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, India.

A. Kavitha
Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, India.

M.A. Gopalan
Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, India.

Correspondence:
M.A. Gopalan
Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, India.

Integral solutions of the heptic equation with five unknowns $x^3 - y^3 + 2(z^3 - w^3) = 3(z - w) + 6(k^2 + 2s^2)T^7$

S. Vidhyalakshmi, A. Kavitha and M.A. Gopalan

Abstract
The non-homogeneous Diophantine equation of degree seven with five unknowns represented by $x^3 - y^3 + 2(z^3 - w^3) = 3(z - w) + 6(k^2 + 2s^2)T^7$ is analyzed for its non-zero distinct integer solutions. A few interesting relation between the solutions and special numbers namely Polygonal numbers, Pyramidal numbers, centered Polygonal numbers are exhibited.

Keywords: Integral solutions, heptic, non-homogeneous equation.
M.sc 2000 mathematics subject classification: 11D41

1. Introduction
Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity as can be seen from [1, 4]. The problem of finding all integer solutions of a diophantine equation with three or more variables and degree at least three, in general presents a good deal of difficulties. There is vast general theory of homogeneous quadratic equations with three variables. Cubic equations in two variables fall into the theory of elliptic curves which is a very developed theory but still an important topic of current research. A lot is known about equations in two variables in higher degrees. For equations with more than three variables and degree at least three very little is known. It is worth to note that undesirability appears in equations, even perhaps at degree four with fairly small co-efficients. It seems that much work has not been done in solving higher order Diophantine equations. In [5-24] a few higher order equations are considered for integral solutions. In this communication a seventh degree non-homogeneous equation, with five variables represented by $x^3 - y^3 + 2(z^3 - w^3) = 3(z - w) + 6(k^2 + 2s^2)T^7$ is considered and in particular a few interesting relations among the solutions are presented.

2. NotationS
$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

CP_n^m : Centered Pyramidal number of rank n with size m.

S_n : Star number of rank n

J_n : Jacobsthal number of rank n

j_n : Jacobsthal-Lucas number of rank n

ky_n : keynea number of rank n.

3. Method of analysis

The non-homogeneous heptic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$x^3 - y^3 + 2(z^3 - w^3) = 3(z - w) + 6(k^2 + 2s^2)T^7 \quad (1)$$

Introduction of the linear transformations,

$$x = u + 1,\ y = u - 1,\ z = v + 1,\ w = v - 1 \quad (2)$$

in (1) leads to $u^2 + 2v^2 = (k^2 + 2s^2)T^7$ (3)

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

Pattern: 3.1

Let $T = a^2 + 2b^2$ (4)

Using (4) in (3) and applying the method of factorization define

$$u + i\sqrt{2}v = (k + i\sqrt{2}s)(a + i\sqrt{2}b)^7 \quad (5)$$

Where

$$\left.\begin{aligned} \alpha &= a^7 - 42a^5b^2 + 140a^3b^4 - 56ab^6 \\ \beta &= 7a^6b - 70a^4b^3 + 84a^2b^5 - 8b^7 \end{aligned}\right\} \quad (6)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= \alpha k - 2\beta s \\ v &= \beta k + \alpha s \end{aligned}\right\} \quad (7)$$

Using (7) and (2) we have

$$\left.\begin{aligned} x(a,b) &= \alpha k - 2\beta s + 1 \\ y(a,b) &= \alpha k - 2\beta s - 1 \\ z(a,b) &= \beta k + \alpha s + 1 \\ w(a,b) &= \beta k + \alpha s - 1 \end{aligned}\right\} \quad (8)$$

Thus, (4) and (8) represent the non-zero distinct integral solutions to (1)

Pattern: 3.2

Consider (3) as

$$u^2 + 2v^2 = (k^2 + 2s^2)T^7 * 1 \quad (9)$$

Write 1 as

$$1 = \frac{(1 + i2\sqrt{2})(1 - i2\sqrt{2})}{9} \quad (10)$$

Substituting (4) and (10) in (9) and employing the factorization method, define

$$u + i\sqrt{2}v = \frac{(1 + i2\sqrt{2})}{3}[(\alpha k - 2\beta s) + \sqrt{2}i(\beta k + \alpha s)]$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} u &= \tfrac{1}{3}[(\alpha k - 2\beta s) - 4(\beta k + \alpha s)] \\ v &= \tfrac{1}{3}[2(\alpha k - 2\beta s) + (\beta k + \alpha s)] \end{aligned}\right\} \quad (11)$$

As our interest is on finding integer solutions, we choose α and β suitably so that u and v are integers. Replace a by 3a and b by 3b in (6). Substituting the corresponding values of α and β in (11) and employing (2) non-zero integral solutions to (1) are found to be

$$\begin{aligned} x(a,b) &= 3^5[(\alpha k - 2\beta s) - 4(\beta k + \alpha s)] + 1 \\ y(a,b) &= 3^5[(\alpha k - 2\beta s) - 4(\beta k + \alpha s)] - 1 \\ z(a,b) &= 3^5[2(\alpha k - 2\beta s) + (\beta k + \alpha s)] + 1 \\ w(a,b) &= 3^5[2(\alpha k - 2\beta s) + (\beta k + \alpha s)] - 1 \\ T(a,b) &= 3^2[a^2 + 2b^2] \end{aligned}$$

For simplicity and clear understanding, we exhibit below the integer solutions and their corresponding properties when

$$k=3 \text{ and } s=1 \quad (12)$$

For this choice, (1) and (3) simplify respectively to,

$$x^3 - y^3 + 2(z^3 - w^3) = 3(z - w) + 66T^7 \quad (13)$$

$$u^2 + 2v^2 = 11T^7 \quad (14)$$

Write 11 as

$$11 = (3 + i\sqrt{2})(3 - i\sqrt{2}) \quad (15)$$

Using (4) and (15) in (14) and applying the method of factorization, define

$$u + i\sqrt{2}v = (3 + i\sqrt{2})(a + i\sqrt{2}b)^7$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= 3\alpha - 2\beta \\ v &= 3\beta + \alpha \end{aligned}\right\} \quad (16)$$

Using (16) in (2) we have,

$$\left.\begin{aligned} x(a,b) &= 3\alpha - 2\beta + 1 \\ y(a,b) &= 3\alpha - 2\beta - 1 \\ z(a,b) &= \alpha + 3\beta + 1 \\ w(a,b) &= \alpha + 3\beta - 1 \end{aligned}\right\} \quad (17)$$

Thus, (4) and (17) represent the integer solution to the equation (13)

Properties

$$(i)\,2[x(1,n)+w(1,n)]+y(1,n)+28t_{4,n}\left(12F_{4,n,6}+6CP_n^6=9t_{4,n}\right)+42t_{4,n}+10=0$$

$$(ii)\,3y(n,1)+2z(n,1)-11w(n,1)=-23t_{4,n}\left[24F_{4,n,3}-12CP_n^3-21t_{4,n}+12\right]+274$$

$$(iii)\,x(n,1)+y(n,1)-3[z(n,1)+w(n,1)]=$$
$$-154t_{4,n}\left[24F_{4,n,3}-12CP_n^3-21t_{4,n}+12\right]+176$$

Note: 3.3

It is seen that in addition to (15), 11 may also be written in two different ways as

$$\left.\begin{aligned}(i)\,11&=\frac{(1+i7\sqrt{2})(1-i7\sqrt{2})}{9}\\(ii)\;\;11&=\frac{(63+i\sqrt{2})(63-i\sqrt{2})}{19^2}\end{aligned}\right\}\qquad(18)$$

Following the procedure similar to the above, the corresponding non-zero integral solutions to the above two cases are as follows:

Case: (i)

$x(a,b)=3^6(\alpha-14\beta)+1$
$y(a,b)=3^6(\alpha-14\beta)-1$
$z(a,b)=3^6(7\alpha+\beta)+1$
$w(a,b)=3^6(7\alpha+\beta)-1$
$T(a,b)=3^2(a^2+2b^2)$

Properties:

$$(i)\,x(1,n)+14w(1,n)=-3^8*154t_{4,n}\left[24F_{4,n,6}+12CP_n^6-18t_{4,n}+3\right]+7217$$

$$(ii)\,z(1,n)-7y(1,n)=3^6*13\left[-7CP_n^{12}+56CP_n^6+4CP_n^3\left(12F_{4,n,4}-3CP_n^{16}-18t_{3,n}-24t_{4,n}\right)\right]-6$$

$$(iii)\,T(2^n,2^n)-3^4J_{2n}$$

Is a cubical integer

Case: (ii)

$x(a,b)=19^6(63\alpha-2\beta)+1$
$y(a,b)=19^6(63\alpha-2\beta)-1$
$z(a,b)=19^6(\alpha+63\beta)+1$
$w(a,b)=19^6(\alpha+63\beta)-1$
$T(a,b)=19^2(a^2+2b^2)$

Properties:

$$(i)\,y(n,1)-63z(n,1)+64=-19^6*3971[7t_{4,n}\left(6F_{4,n,6}+3CP_n^6-12t_{4,n}+12\right)-8$$

$$(ii)\,T(2^n,2^{n+1})=19[KY_n+J_{2n+3}-2(3J_n+(-1)^n)]$$

$$(iii)\,63x(1,n)+2w(1,n)=3971*19^6[1-14t_{4,n}\left(6F_{4,n,6}+12CP_n^6+2t_{4,n}+3\right)]$$

Note: 3.4

Consider (13) as

$$x^3-y^3+2(z^3-w^3)=[3(z-w)+66T^2]*1\qquad(19)$$

Factorize 1 as

$$1=\frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{9}\qquad(20)$$

Substituting (20) in (19) and following the analysis presented above as in Pattern.2, the corresponding integer solutions to (13) are given by

$x(a,b)=-3^6(\alpha+14\beta)+1$
$y(a,b)=-3^6(\alpha+14\beta)-1$
$z(a,b)=3^6(7\alpha-\beta)+1$
$w(a,b)=3^6(7\alpha-\beta)-1$
$T(a,b)=3^2(a^2+2b^2)$

It is to be noted that, in addition to (20), 1 is also written as

$$1 = \frac{(7+i6\sqrt{2})(7-i6\sqrt{2})}{121}$$

For this choice, the integer solutions to (13) are obtained as

$$x(a,b) = 11^6(9\alpha - 50\beta) + 1$$
$$y(a,b) = 11^6(9\alpha - 50\beta) - 1$$
$$z(a,b) = 11^6(25\alpha + 9\beta) + 1$$
$$w(a,b) = 11^6(25\alpha + 9\beta) - 1$$
$$T(a,b) = 11^2(a^2 + 2b^2)$$

4. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous heptic equation with five unknowns. As the heptic equations are rich in variety, one may search for other forms of heptic equation with variables greater than or equal to five and obtain their corresponding properties.

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