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On the ternary quadratic diophantine equation $3(x^2+y^2)-4xy=18z^2$

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Abstract
The ternary quadratic diophantine equation represented by $3(x^2+y^2)-4xy=18z^2$ is analyzed for its non-zero distinct integer solutions. A few interesting properties between the solutions and special figurate numbers are obtained.

Keywords: Ternary quadratic, integer solutions, figurate numbers.

2010 Mathematics subject classification: 11D09.

Notations Used:

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

$$SO_n=n(2n^2-1)$$

$$Pr_n=n(n+1)$$

$$OH_n=\frac{n(2n^2+1)}{3}$$

$$Ct_{m,n}=\frac{mn(n+1)+2}{2}$$

$$CP_n^m=\frac{m(n-1)n(n+1)}{6}+n$$

$$S_n=6n(n-1)+1$$

$$Pt_n=\frac{n(n+1)(n+2)(n+3)}{24}$$

$$g_{na}=2a-1$$

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-21] for cubic equations with three unknowns. This communication concerns with yet another interesting equation $3(x^2+y^2)-4xy=18z^2$ representing non-homogeneous Quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. Method of Analysis

The ternary quadratic Diophantine equation to be solved is

$$3(x^2+y^2)-4xy=18z^2 \tag{1}$$

Introducing the linear transformations

$$x = u + v \; ; \; y = u - v \tag{2}$$

In (1), it is written as

$$u^2 + 5v^2 = 9z^2 \tag{3}$$

Which is satisfied by

$$u = 5p^2 - q^2$$

$$v = 2pq$$

$$z = \frac{5p^2 + q^2}{3}$$

Where p and q are non-zero distinct numbers.

Replacing p by 3P & q by 3Q in the above equations and using (2), the non-zero distinct integral solutions of (2) in two parameters are given by

$$x = x(P, Q) = 45P^2 - 9Q^2 + 18PQ$$

$$y = y(P, Q) = 45P^2 - 9Q^2 - 18PQ$$

$$z = z(P,Q) = 15P^2 + 3Q^2$$

A few interesting properties are as follows

Properties

- $x(n,1) + y(n,1) + 6z(n,1) = 180\, t_{2,n}^2$
- $x(\Pr_n,1) + 3z(\Pr_n,1) = 90\, \Pr_n^2 + 18\, \Pr_n$
- $x(a,a) - y(a,a) = 36a^2 =$ Perfect square
- $3z(n^2,n) - y(n^2,n) = 18t_{2,n}^2 + 18CP_{6,n}$
- $x(n,1) + y(n,1) + 6z(n,1) - 150 - 30S_n = 0$

It is noted that (3) may be solved through different methods leading to different patterns of solutions to (1) which are illustrated below.

Pattern: 1

Write 9 as

$$9 = \left(2 + i\sqrt{5}\right)\left(2 - i\sqrt{5}\right) \tag{4}$$

Assume $z = a^2 + 5b^2 = \left(a + i\sqrt{5}b\right)\left(a - i\sqrt{5}b\right)$ (5)

Where a, b are non-zero distinct integers.

Using (4) & (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{5}v = \left(2 + i\sqrt{5}\right)\left(a + i\sqrt{5}b\right)^2 \tag{6}$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = 2a^2 - 10b^2 - 10ab$$

$$v = v(a,b) = a^2 - 5b^2 + 4ab$$

Substituting the above values of u & v in equation (2), the values of x and y are given by

$$x = x(a,b) = 3a^2 - 15b^2 - 6ab \tag{7}$$

$$y = y(a,b) = a^2 - 5b^2 - 14ab \tag{8}$$

Thus (5), (7) & (8) represents a non-zero distinct integral solutions of (1) in two parameters.

Properties

- $x(n,n+1) - 3y(n,n+1) - 36\Pr_n = 0$
- $3z(a,a) - x(a,a) - 30t_{4,a} =$ Nasty number
- $7x(a,b) - 3y(a,b) + 18z(a,b) =$ Perfect square
- $x(n^2,n+1) - 3y(n^2,n+1) - 72P_n^5 = 0$
- $3z(n+1,n) - x(n+1,n) = 6\Pr_n + 30t_{3,n}^2$

Pattern: 2

Write 9 as

$$9 = \left(-2 + i\sqrt{5}\right)\left(-2 - i\sqrt{5}\right) \tag{9}$$

Using (5) & (9) in (3) and applying the method of factorization, define

$$u + i\sqrt{5}v = \left(-2 + i\sqrt{5}\right)\left(a + i\sqrt{5}b\right)^2 \tag{10}$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = -2a^2 + 10b^2 - 10ab$$

$$v = v(a,b) = a^2 - 5b^2 - 4ab$$

Substituting the above values of u & v in equation (2), the values of x and y are given by

$$x = x(a,b) = -a^2 + 5b^2 - 14ab \tag{11}$$

$$y = y(a,b) = -3a^2 + 15b^2 - 6ab \tag{12}$$

Thus (5), (11) & (12) represents a non-zero distinct integral solutions of (1) in two parameters.

Properties

- $3x(1,\Pr_n) - y(1,\Pr_n) + 36\Pr_n = 0$
- $3z(a,b) - y(a,b) - 6ab =$ Nasty number
- $6x(a,b) - 14y(a,b) + 36z(a,b) = 72a^2$
- $y(n,2n^2+1) - 3x(n,2n^2+1) = 108OH_n$
- $3z(n,n+1) - y(n,n+1) = S_n + 12t_{3,n} + 5$

Pattern: 3

Instead of (5), we write z as

$$z = a^2 + 5b^2 = \left(-a + i\sqrt{5}b\right)\left(-a - i\sqrt{5}b\right) \tag{13}$$

Using (9) & (13) in (3) and applying the method of factorization, define

$$u + i\sqrt{5}v = \left(-2 + i\sqrt{5}\right)\left(-a + i\sqrt{5}b\right)^2 \tag{14}$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = -2a^2 + 10b^2 + 10ab$$
$$v = v(a,b) = a^2 - 5b^2 + 4ab$$

Substituting the above values of u & v in equation (2), the values of x and y are given by

$$x = x(a,b) = -a^2 + 5b^2 + 14ab \quad (15)$$
$$y = y(a,b) = -3a^2 + 15b^2 + 6ab \quad (16)$$

Thus (13), (15) & (16) represents a non-zero distinct integral solutions of (1) in two parameters.

Properties

- $x(1,1) + z(1,1) + 1 = CS_4$
- $3z(a,a) + y(a,a) - 30a^2 = Nastynumber$
- $3x(1,1) - y(1,1) = Pt_4 + 1$
- $x(1,\Pr_n) + z(1,\Pr_n) = 10\Pr_n^2 + 14\Pr_n$
- $3x(n,n) - y(n,n) = 36t_{3,n}^2$

Pattern: 4

Write (3) in the form of ratio as,

$$\frac{u+2z}{z-v} = \frac{5(z+v)}{u-2z} = \frac{p}{q}, \; q \neq 0$$

Which is equivalent to the following two equations

$$(2q - p)z + qu + pv = 0$$
$$(5q + 2p)z - pu + 5qv = 0$$

On employing the method of cross multiplication, we get

$$u = u(p,q) = 2p^2 - 10q^2 + 10pq$$
$$v = v(p,q) = p^2 - 5q^2 - 4pq$$
$$z = z(p,q) = p^2 + 5q^2 \quad (17)$$

Substitute the values of u and v in (2), the values of x and y are given by

$$x = x(p,q) = 3p^2 - 15q^2 + 6pq \quad (18)$$
$$y = y(p,q) = p^2 - 5q^2 + 14pq \quad (19)$$

Thus (17), (18) & (19) represents non-zero distinct integral solutions of (1) in two parameters.

Properties

- $x(n,2n^2-1) - 3y(n,2n^2-1) = 36SO_n$
- $3z(n^2,n) - x(n^2,n) - 30t_{2,n}^2 + 6CP_{6,n} = 0$
- $x(n,n+1) - 3y(n,n+1) = 36\Pr_n$
- $y(n,n+1) + z(n,n+1) - 2t_{2,n}^2 - 14\Pr_n = 0$
- $14x[n(n+1),1] - 6y[n(n+1),1] + 36z[n(n+1),1] = 72\Pr_n^2$

Note

(3) Can also be expressed in the form of ratio

$$\frac{u+2z}{5(z-v)} = \frac{z+v}{u-2z} = \frac{p}{q}, q \neq 0$$

Repeating the analysis as above the corresponding integer solutions along with properties are presented below.

The solution is

$$x = x(p,q) = 15p^2 - 3q^2 + 6pq$$
$$y = y(p,q) = 5p^2 - q^2 + 14pq$$
$$z = z(p,q) = 5p^2 + q^2$$

Properties

- $3y(a,a) - x(a,a) = 36a^2 =$ Perfect square
- $y(n,n) + z(n,n) - 24t_{2,n}^2 = 0$
- $x(n,n+1) + 3z(n,n+1) = 30t_{2,n}^2 + 6\Pr_n$
- $x(n,2n^2-1) - 3y(n,2n^2-1) + 36SO_n = 0$
- $x(n,n) + 3z(n,n) - 30t_{2,n}^2 =$ Nasty number

Pattern: 5

Consider (3), as

$$u^2 + 5v^2 = 9z^2 = 9z^2 \times 1 \quad (20)$$

Write 1 as, $1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9}$ (21)

Using (4), (5) & (21) in (20) and applying the method of factorization, define

$$u + i\sqrt{5}v = (2+i\sqrt{5})(a+i\sqrt{5}b)^2 \frac{(2+i\sqrt{5})}{3}$$

Equating the real and imaginary parts and replacing a by 3A & b by 3B, we have

$$u = u(A,B) = -3A^2 + 15B^2 - 120AB$$
$$v = v(A,B) = 12A^2 - 60B^2 - 6AB$$

Also, (5) $\Rightarrow z = z(A,B) = 9(A^2 + 5B^2)$ (22)

Substituting the above values of u & v in (2), the values of x and y are given by

$$x = x(A,B) = 9A^2 - 45B^2 - 126AB \quad (23)$$
$$y = y(A,B) = -15A^2 + 75B^2 - 114AB \quad (24)$$

Thus (23), (24) & (22) represents the non-zero distinct integral solutions of (1) in two parameters.

Properties

- $5x(1,1) + 3y(1,1) + 972t_{3,1} = 0$
- $x[1,n(n+1)] + z[1,n(n+1)] = 18t_{3,1} - 126\Pr_n$
- $z(\Pr_n,1) - x(\Pr_n,1) - 90t_{3,1} - 126\Pr_n = 0$
- $3y(n,n) + 5x(n,n) + 810 + 162S_n = 0$
- $x(1,n) + z(1,n) = 18t_{3,1} - 126t_{2,n}$

Note
Instead of (21), one may also consider 1 as

$$1 = \frac{\left(-2 + i\sqrt{5}\right)\left(-2 - i\sqrt{5}\right)}{9} \quad (25)$$

Substituting (4), (5) & (25) in (20) and following the procedure presented above, the corresponding integral solutions of (1) are given by

$$x = x(a,b) = -3a^2 + 15b^2 + 6ab$$

$$y = y(a,b) = -3a^2 + 15b^2 - 6ab$$

Properties

- $x(1, Pr_n) + y(1, Pr_n) + 9z(1, Pr_n) = 75\ Pr_n^2$
- $x(1, n) + y(1, n) - 30t_{2,n}^2 + 9t_{3,n}^2 = 0$
- $3z(a,a) - y(a,a) - 9a^2 =$ Nasty number
- $x[(1, n(2n^2+1)] + y[(1, n(2n^2+1)] + 9z[(1, n(2n^2+1)] = 675OH_n^2$
- $x[(1, n(n+1)] + y[(1, n(n+1)] + 9t_{1,2} - 30\,Pr_n^2 = 0$

Pattern: 6
In addition to (25), consider 1 as

$$1 = \frac{\left(2 + i3\sqrt{5}\right)\left(2 - i3\sqrt{5}\right)}{49} \quad (26)$$

Substituting (5), (9) & (26) in (20) and following the procedure presented above, the corresponding integral solutions of (1) are given by

$$x = x(A,B) = -21A^2 + 105B^2 - 714AB$$

$$y = y(A,B) = -133A^2 + 665B^2 - 406AB$$

$$z = z(A,B) = 49\left(A^2 + 5B^2\right)$$

Properties

- $21x(1, Pr_n) + 9z(1, Pr_n) = 2450\ Pr_n^2 - 14994\ Pr_n$
- $3y(n, n+1) - 19x(n, n+1) = 12348\,Pr_n$
- $19z(n, n(n+1)) - 7y(n, n(n+1)) = 1862t_{2,n}^2 + 5684P_n^5$
- $3y(n, 2n^2-1) - 19x(n, 2n^2-1) = 12348SO_n$
- $21x(n+1, n) + 9z(n+1, n) = 2450t_{2,n}^2 - 14994\,Pr_n$

Note
Instead of (26), we write 1 as

$$1 = \frac{\left(-2 + i3\sqrt{5}\right)\left(-2 - i3\sqrt{5}\right)}{49}$$

Following the procedure presented above, the corresponding integral solutions of (1) are given by

$$x = x(A,B) = -105A^2 + 525B^2 - 546AB$$

$$y = y(A,B) = -161A^2 + 805B^2 - 14AB$$

$$z = z(A,B) = 49\left(A^2 + 5B^2\right)$$

Properties

- $x(1, Pr_n) - 39y(1, Pr_n) = 6174t_{3,1}^2 - 30870\ Pr_n^2$
- $x(1,1) - 39y(1,1) + 126z(1,1) = 98Pt_6$
- $7y(n^2,1) + 23z(n^2,1) = 322\,Pt_4 - 98t_{2,n}^2$
- $x(n, 2n^2+1) - 39y(n, 2n^2+1) = 6174t_{2,n}^2 - 277830OH_n^2$
- $7y(2n^2-1, n) + 23z(2n^2-1, n) = 11270\,t_{2,n}^2 + 98SO_n$

Pattern: 7
Equation (3) can be written as

$$u^2 = 9z^2 - 5v^2 \quad (27)$$

Introducing the linear transformations,

$$z = X + 5T \ \&\ v = X + 9T \quad (28)$$

In (27), we get

$$4X^2 - u^2 = 180T^2 \quad (29)$$

Which is satisfied by

$$T(\alpha,\beta) = -4\alpha\beta \quad (30)$$

$$X(\alpha,\beta) = -12\alpha^2 - 15\beta^2 \quad (31)$$

$$u(\alpha,\beta) = -24\alpha^2 + 30\beta^2 \quad (32)$$

$$v(\alpha,\beta) = -12\alpha^2 - 15\beta^2 - 36\alpha\beta$$

Substituting the values of (30) & (31) in (28) and using (2), the corresponding integer solutions of (1) are given by

$$x = x(\alpha,\beta) = -36\alpha^2 + 15\beta^2 - 36\alpha\beta$$

$$y = y(\alpha,\beta) = -12\alpha^2 + 45\beta^2 + 36\alpha\beta$$

$$z = z(\alpha,\beta) = -12\alpha^2 - 15\beta^2 - 20\alpha\beta$$

Properties

- $3y[(2n^2-1), n] - x[(2n^2-1), n] - 100 = 20S_n + 144OH_n$
- $y[n(n+1), (n+2)] - 3x[n(n+1), (n+2)] = 96Pr_n^2 + 864P_n^3$
- $y(n^2, n) - z(n^2, n) = 60t_{2,n}^2 + 56CP_{6,n}$
- $x(1,1) + y(1,1) + 1 = CS_3$
- $x(Pr_n, 1) + y(Pr_n, 1) - 4z(Pr_n, 1) = 24CS_2 + 80\,Pr_n$

Note
In addition to (28), one may also consider the linear transformations $z = X - 5T$ & $v = X - 9T$. Following the method presented above, different set of solutions are obtained.

3. Conclusion
In this paper, we have obtained infinetly many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$3(x^2 + y^2) - 4xy = 18z^2$$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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