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On The Ternary Quadratic Diophantine Equation $3(x^2 + y^2) – 5xy = 60z^2$

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Abstract

The ternary quadratic diophantine equation represented by $3(x^2+y^2)-5xy=60z^2$ is analyzed for its non-zero distinct integer solutions. A few interesting properties between the solutions and special figurate numbers are obtained.

Keywords: Ternary quadratic, integer solutions, figurate numbers.

2010 Mathematics subject classification: 11D09.

Notations Used:

$$t_{m,n} = n\left(1+\frac{(n-1)(m-2)}{2}\right)$$

$$Pr_n = n(n+1)$$

$$S_n = 6n(n-1)+1$$

1. Introduction

The Diophantine equations offer an unlimited field for research due to their variety [13].In particular, one may refer [4-21] for quadratic equations with three unknowns.This communication concerns with yet another interesting equation $3(x^2+y^2)-5xy=60z^2$ representing homogeneous quadractic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solution are presented.

2. Method of Analysis

The ternary quadratic Diophantine equation to be solved is

$$3(x^2+y^2)-5xy=60z^2 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following non-zero integer's triples (x,y,z): (-14588,-17556,2170), (12486,10082,1545), (21122,16566,2621), (5394,4742,669), (3802,3534,477), (2534,2562,329), (1342,1626,201), (1218,1526,189), (606,1018,137)

As the considered equation is symmetric in x, y and z, we have presented only positive integer solutions for clear understanding.

However, we have other choices of solutions to (1) which are illustrated bellow:

The substitution of the linear transformations

$$x = u+v;\; y = u-v \quad (u \neq 0, v \neq 0) \tag{2}$$

in (1) leads to

$$u^2 + 11v^2 = 60z^2 \tag{3}$$

Take

$$z = z(a, b) = a^2 + 11b^2 \tag{4}$$

where a, b are non-zero distinct integers. Different patterns of solutions of (1) are illustrated below

Pattern: 1

Write 60 as

$$60=(7+i\sqrt{11})(7-i\sqrt{11}) \quad (5)$$

Substituting (4), (5) in (3) and employing the method of factorization, we've

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v)=(7+i\sqrt{11})$$

$$*(7-i\sqrt{11})(a+i\sqrt{11}b)^2(a-i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{11}v)=(7+i\sqrt{11})(a+i\sqrt{11}b)^2 \quad (6)$$

$$(u-i\sqrt{11}v)=(7-i\sqrt{11})(a-i\sqrt{11}b)^2 \quad (7)$$

Equating the real and imaginary parts in (6)

$$u=u(a,b)=7a^2-22ab-77b^2$$

$$v=v(a,b)=a^2+14ab-11b^2$$

Substituting the values of u and v in (2), we've

$$x=x(a,b)=8a^2-8ab-88b^2 \quad (8)$$

$$y=y(a,b)=6a^2-36ab-66b^2 \quad (9)$$

Thus (8), (9) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties

- $x(2a,1)-32Pr_a \equiv -88 \pmod{48}$
- $y(b,1)-6Pr_a \equiv -66 \pmod{42}$
- $x(1,2b)+352Pr_a \equiv 8 \pmod{336}$
- $[5z(a,-a)-2z(a,-a)]$ is a perfect square
- $x(a,1)-y(a,1)+z(a,1)-3Pr_a \equiv -11 \pmod{25}$

Pattern: 2

Write 60 as

$$60=(4+i2\sqrt{11})(4-i2\sqrt{11}) \quad (10)$$

Substituting (4), (10) in (3) and employing the method of factorization, we've

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v)=(4+i2\sqrt{11})$$

$$*(4-i2\sqrt{11})(a+i\sqrt{11}b)^2(a-i\sqrt{11}b)^2$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{11}v)=(4+i2\sqrt{11})(a+i\sqrt{11}b)^2 \quad (11)$$

$$(u-i\sqrt{11}v)=(4-i2\sqrt{11})(a-i\sqrt{11}b)^2 \quad (12)$$

Equating the real and imaginary parts in (11)

$$u=u(a,b)=4a^2-44ab-44b^2$$

$$v=v(a,b)=2a^2+8ab-22b^2$$

Substituting the values of u and v in (2), we've

$$x=x(a,b)=6a^2-36ab-66b^2 \quad (13)$$

$$y=y(a,b)=2a^2-52ab-22b^2 \quad (14)$$

Thus (13), (14) and (4) represent non-zero distinct integral solutions of (1) in two parameters.

Properties

- $x(2a,1)-24pr_a \equiv -66 \pmod{96}$
- $x(1,b)+66pr_b \equiv 6 \pmod{30}$
- $x(1,2b)+264pr_b \equiv 6 \pmod{192}$
- $[6z(b,-b)-3z(b,-b)]$ is a perfect square
- $x(a,1)-y(a,1)+z(a,1)-5Pr_a \equiv -33 \pmod{21}$

Pattern: 3

Write (1) as

$$3(x^2+y^2)-5xy=60z^2$$

$$=60z^2*1 \quad (15)$$

$$1=\frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100} \quad (16)$$

Substituting (4), (5), (16) in (15), it is written in the factorizable form as

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v)=(7+i\sqrt{11})(7-i\sqrt{11})$$

$$*(a+i\sqrt{11}b)^2(a-i\sqrt{11}b)^2*\frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100}$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{11}v)=(7+i\sqrt{11})(a+i\sqrt{11}b)^2$$

$$*\frac{(1+i3\sqrt{11})}{10} \quad (17)$$

$$(u-i\sqrt{11}v)=(7-i\sqrt{11})(a-i\sqrt{11}b)^2$$

$$*\frac{(1-i3\sqrt{11})}{10} \quad (18)$$

Equating the real and imaginary parts in (17)

$$u=\frac{1}{10}(-26a^2-484ab+286b^2) \quad (19)$$

$$v=\frac{1}{10}(22a^2-52ab-242b^2) \quad (20)$$

Replacing a by 10A, b by 10B in (19), (20) and (4), we've

$$u=u(A,B)=-260A^2-4840AB+2860B^2$$

$$v=v(A,B)=220A^2-520AB-2420B^2$$

$$z=z(A,B)=100A^2+1100B^2 \quad (21)$$

Substituting the values of u and v in (2), we've

$$x=x(A,B)=-40A^2-5360AB$$

$$+440B^2 \quad (22)$$

$$y=y(A,B)=-480A^2-4320AB$$

$$+5280B^2 \quad (23)$$

Thus (22), (23) and (21) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:
- ❖ $x(a,1)+40Pr_a \equiv 440 \pmod{5320}$
- ❖ $y(2,-a)-5280\,pr_a \equiv -1920 \pmod{3360}$
- ❖ $x(a,1)+y(a,1)-z(a,1)+620\,pr_a \equiv -4620 \pmod{9060}$
- ❖ $y(1,-2a)-21120pr_a \equiv -480 \pmod{12480}$
- ❖ $[9z(b,b)-6z(b,b)]$ is a perfect square

Pattern: 4

Write 1 as

$$1=\frac{(-1+i3\sqrt{11})(-1-i3\sqrt{11})}{100} \quad (24)$$

Substituting (4), (5), (24) in (15), it is written in the factorizable form as

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v)=(7+i\sqrt{11})(7-i\sqrt{11}) * (a+i\sqrt{11}b)^2(a-i\sqrt{11}\,b)^2 * \frac{(-1+i3\sqrt{11})(-1-i3\sqrt{11})}{100}$$

Equating the positive and negative factors, we get

$$(u+i\sqrt{11}v)=(7+i\sqrt{11})(a+i\sqrt{11}b)^2 * \frac{(-1+i3\sqrt{11})}{10} \quad (25)$$

$$(u-i\sqrt{11}v)=(7-i\sqrt{11})(a-i\sqrt{11}b)^2 * \frac{(-1-i3\sqrt{11})}{10} \quad (26)$$

Equating the real and imaginary parts in (25)

$$u=\frac{1}{10}(-40a^2-440ab+440\,b^2) \quad (27)$$

$$v=\frac{1}{10}(20a^2-80ab-220b^2) \quad (28)$$

Replace a by 10A, b by 10B in (27), (28) and (4)

$$u=u(A,B)=-400A^2-4400AB+4400B^2$$

$$v=v(A,B)=200A^2-800AB-2200B^2$$

$$z=z(A,B)=400\,A^2+1100B^2 \quad (29)$$

Substituting the values of u and v in (2), we've

$$x=x(A,B)=-200A^2-5200AB+2200\,B^2 \quad (30)$$

$$y=y(A,B)=-600A^2-3600AB+6600\,B^2 \quad (31)$$

Thus (30), (31) and (29) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:
- ❖ $x(1,-a)-2200pr_a \equiv -200 \pmod{3000}$
- ❖ $y(1,-b)-6600\,pr_b \equiv -600 \pmod{3000}$
- ❖ $x(b,1)-y(b,1)+z(b,1)-800\,pr_b \equiv -5500 \pmod{2400}$
- ❖ $y(1,-2a)-26400pr_a \equiv -600 \pmod{19200}$
- ❖ $[3z(b,-b)-10z(b,-b)]$ is a perfect square

Pattern: 5

The ternary quadratic equation (3) can be written as

$$(u+7z)(u-7z)=11(z+v)(z-v) \quad (32)$$

The above equation is written in the form of ratio as

$$\frac{(u+7z)}{(z-v)}=\frac{11(z+v)}{(u-7z)}=\frac{a}{b} \quad (b\neq 0) \quad (33)$$

The equation (33) is equivalent to the following two equations

$$bu+av+z(7b-a)=0 \quad (34)$$

$$au+z(-7a-11b)-11vb=0 \quad (35)$$

Applying the method of cross multiplication, we get,

$$\frac{u}{-7a^2+77b^2-22ab}=\frac{v}{-a^2+11b^2+14ab}=\frac{z}{-(a^2+11b^2)}$$

Therefore,

$$u=u(a,b)=-7a^2+77b^2-22ab$$

$$v=v(a,b)=-a^2+11b^2+14ab$$

$$z=z(a,b)=-(a^2+11b^2) \quad (36)$$

Substituting in the values of u and v in (2), we've

$$x=x(a,b)=-8a^2-8ab+88b^2 \quad (37)$$

$$y=y(a,b)=-6a^2-36ab+66b^2 \quad (38)$$

Thus (37), (38) and (36) represent non-zero distinct integral solutions of (1) in two parameters.

Properties
- ❖ $x(2b,1)+32Pr_b \equiv 88 \pmod{16}$
- ❖ $y(1,-2b)-264Pr_b \equiv -6 \pmod{192}$
- ❖ $x(-1,2a)-352Pr_a \equiv -8 \pmod{336}$
- ❖ $x(b,1)-y(b,1)+z(b,1)+3Pr_b \equiv 33 \pmod{25}$
- ❖ $x(2,a)+66Pr_a \equiv 24 \pmod{6}$

Remark

In addition to (33), (3) may also be expressed in the form of ratios in two different ways that are presented below:

WAY 1:

$$\frac{(u+7z)}{11(z-v)}=\frac{(z+v)}{(u-7z)}=\frac{a}{b}$$

WAY 2:

$$\frac{(u+7z)}{11(z+v)}=\frac{(z-v)}{(u-7z)}=\frac{a}{b}$$

Solving each of the above system of equations by following the procedure presented in pattern 5, the corresponding integer solutions to (1) are found to be as given below:

Solution for way 1:

$$x = x(a,b) = 88a^2 + 8ab - 8b^2$$
$$y = y(a,b) = 66a^2 + 36ab - 6b^2$$
$$z = z(a,b) = 11a^2 + b^2$$

Solution for way 2:

$$x = x(a,b) = -66a^2 - 36ab + 6b^2$$
$$y = y(a,b) = -88a^2 - 8ab + 8b^2$$
$$z = z(a,b) = -11a^2 - b^2$$

Pattern: 6

Write equation (3) as

$$u^2 = 60z^2 - 11v^2 \quad (*)$$

Substituting the linear transformations

$$z = X - 11T$$
$$v = X - 60T \quad (**)$$

in (*) we get,

$$u^2 = 49X^2 - 32340T^2 \quad (39)$$

which is satisfied by

$$T = 2pq \quad (40)$$
$$u = 32340p^2 - q^2 \quad (41)$$
$$x = 4620p^2 + \frac{q^2}{7} \quad (42)$$

Replace 'p' by 'P' and 'q' by '7Q' in (40), (41), (42)

$$T = 14PQ \quad (43)$$
$$u = 32340P^2 - 49Q^2 \quad (44)$$
$$X = 4620P^2 + 7Q^2 \quad (45)$$

From (**), we ge

$$z = 4620P^2 + 7Q^2 - 154PQ \quad (46)$$
$$v = 4620P^2 + 7Q^2 - 840PQ \quad (47)$$

Substituting the values of u and v in (2), we've

$$x = x(P,Q) = 36960P^2 - 42Q^2 - 840PQ \quad (48)$$
$$y = y(P,Q) = 27720P^2 - 56Q^2 + 840PQ \quad (49)$$

Thus (48), (49) and (46) represent non-zero distinct integral solutions of (1) in two parameters.

Properties

- ❖ $x(q,-2) - 36960Pr_q \equiv -168 \pmod{35280}$
- ❖ $y(q,2) - 27720Pr_q \equiv -224 \pmod{26040}$
- ❖ $x(q,1) + y(q,1) - z(q,1) - 60060Pr_q \equiv -105 \pmod{59906}$
- ❖ $x(2q,-2) - 147840Pr_q \equiv -168 \pmod{144480}$
- ❖ $x(2q,-2) + y(2q,-2) + z(2q,-2) - 27720Pr_q \equiv -364 \pmod{27104}$

3. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by

$$3(x^2 + y^2) - 5xy = 60z^2$$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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