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On Homogeneous Ternary quadratic Diophantine Equation $a(x^2+y^2)-bxy=4az^2, b\neq 2a$

$a(x^2+y^2)-bxy=4az^2, b\neq 2a$

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Abstract
The ternary quadratic homogeneous equation representing homogeneous cone given by $a(x^2+y^2)-bxy=4az^2, b\neq 2a$ is analyzed for its non-zero distinct integer points on it. Six different patterns of integer points satisfying the cone under consideration are obtained. A few interesting properties among the solutions and polygonal numbers are presented.

Keywords: Ternary homogeneous quadratic, integral solutions.
2010 Mathematics subject classification: 11D09.

Notation Used:

$t_{m,n}$ – Polygonal number of rank n with size m.

P_n^m – Pyramidal number of rank n with size m.

1. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 21]. For an extensive review of various problems, one may refer [2-20, 22]. This communication concerns with yet another interesting ternary quadratic equation $a(x^2+y^2)-bxy=4az^2, b\neq 2a$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented. well. IQ is an important measurement, but only if it is carried out alongside other assessment and measurement including social functioning and adaptation.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero distinct integer solution is

$a(x^2+y^2)-bxy=4az^2$, b ≠ ±2a (1) Note that (1) is satisfied by (0, ±2a, z), (±2a, bz, z). However, we have the other choices of solutions which are illustrated below. The substitution of the linear transformations

$x = U + V, y = U - V$ (2)

in (1) leads to

$(2a - b) U^2 + (2a + b) V^2 = 4az^2$ (3)

Again introducing the transformations

$$\left.\begin{aligned} U &= X + (2a + b) T, \\ V &= X - (2a - b) T \end{aligned}\right\} \quad (4)$$

in (3), it leads to

$X^2 + (4a^2 - b^2) T^2 = z^2$ (5)

Now, (5) is solved through different methods and thus, different patterns of solutions to (1) are obtained. s

Method: 1

Assume that $(4a^2 - b^2)$ is not a perfect square.

(5) is written in the factorizable form as $(z + X)(z - X) = (2a + b)(2a - b) T^2$

Write the above equation in the form of ratio as

$$\frac{z+X}{(2a+b)T}=\frac{(2a-b)T}{z-X}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equations.

$\beta X + \beta z - \alpha (2a + b) T = 0$
$\alpha X - \alpha z + \beta (2a - b) = 0$, where $\alpha, \beta \neq 0$.

Applying the method of cross multiplication we get,

$X = \alpha^2 (2a + b) - \beta^2 (2a - b)$
$T = 2\alpha\beta$
$Z = \alpha^2 (2a + b) - \beta^2 (2a - b)$ (6)

Substituting the values of X and T in (4) and using (2), the values of x and y are given by

$x = x (a, b, \alpha, \beta) = 2\alpha^2 (2a + b) - 2\beta^2 (2a - b) + 4b\alpha\beta$
$y = y (a, b, \alpha, \beta) = 8a\alpha\beta$ (7)

Thus (6) & (7) represent the integer solutions to (1).

Properties

1. $x (a, a+1, a+2, a+3) - 3C P_a^{16} - t_{82, a} \equiv 50 \pmod{114}$, where $a \neq 1$
2. $x (a, a, a, a) - 2 z (a, a, a, a) + y (a, a, a, a) - 8 a^3 = 0$
3. $y\left(\frac{\alpha(\alpha+1)}{2}, b, \alpha, 1\right) - 8 P_\alpha^5 = 0$.
4. $x (1, 1, 1, \alpha) + t_{6,\alpha} \equiv 0 \pmod 3$.

Method 2

Rewrite (5) as
$X^2 - b^2 T^2 = z^2 - 4a^2 T^2$ (8)
Factorizing (8) we've
$(X + b T) (X - b T) = (z + 2a T) (z - 2a T)$

Write the above equation in the form of ratio as

$$\frac{X+bT}{z+2aT}=\frac{z-2aT}{X-bT}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the system of double equations.
$\beta x - \alpha z + (b\beta - 2a\alpha) T = 0$
$\alpha x - \beta z - (\alpha b - 2a\beta) T = 0$
Applying the method of cross multiplication, we get

$X = b (\alpha^2 + \beta^2) - 4a\alpha\beta$
$T = \alpha^2 - \beta^2$
$Z = 2b\alpha\beta - 2a (\alpha^2 + \beta^2)$ (9)

Substituting the values of X and T in (4) and using (2), the values of x and y are given by

$x = x (a, b, \alpha, \beta) = 4b\alpha^2 - 8a\alpha\beta$
$y = y (a, b, \alpha, \beta) = 4a (\alpha^2 - \beta^2)$ (10)

Thus (9) and (10) represent the integer solutions to (1).

Properties

1. $x(1, 1, \alpha, 1) - 20 t_{3, \alpha} \equiv - 2 \pmod{20}$
2. $z(2, 2, \alpha, 2) + 8 t_{3, \alpha} \equiv - 4 \pmod{20}$
3. $x(a, 1, 1, 1) + y(a, 1, 1, 1) + z(a, 1, 1, 1) \equiv 6 \pmod{12}$
4. $x(1, 1, \alpha, 1) + y(1, 1, \alpha, 1) + z(1, 1, \alpha, 1) - t_{14, \alpha} + \alpha \equiv - 6$
5. $y(a, 1, a, 1) - 8 P_a^5 + 8 t_{3, a} = 0$

Method: 3

Assume $z = z (a, b, p, q)$
$= p^2 + (4a^2 - b^2) q^2$ (11)
where $(4a^2 - b^2)$ is not a perfect square. Substituting (11) in (5) and employing the method of factorization, define

$$(X + i\sqrt{4a^2-b^2}\,T) = \left(p + i\sqrt{4a^2-b^2}\,q\right)^2$$

Equating real and imaginary parts, we've

$X = p^2 - (4a^2 - b^2) q^2$
$T = 2pq$
Substituting X and T in (4) and using (2) the values of x and y are given by

$x = x (a, b, p, q)$
$= 2p^2 - 2q^2 (4a^2 - b^2) q^2$
$y = y (a, b, p, q) = 8apq$ (12)

Thus (11) and (12) represent the integer solutions to (1).

Properties

1. $y (a, b, a (a + 1), 1) - 16 P_a^5 = 0$
2. $x (1, 1, p, p) + y (1, 1, p, p) + z (1, 1, p, p) - t_{26, p} \equiv 0 \pmod{11}$
3. $x (2, 2, p, p) + y (2, 2, p, p) + t_{8, p} \equiv 0 \pmod 2$
4. $y (a, a, (a+1), (a+2)) - 48 P_a^3 = 0$
5. $z (a, a+1, 1, 1) - t_{8, a} = 0$, where $a \neq 1$.

Method: 4

One may write (5) as
$X^2 + (4a^2 - b^2) T^2 = z^2 = z^2 * 1$ (13)

Assume $z = z(a, b, q, q)$
$= (p^2 + (4a^2 - b^2) q^2) (4a^2)$ (14)

Write (1) as

$$1 = \frac{\left(b+i\sqrt{4a^2-b^2}\right)\left(b-i\sqrt{4a^2-b^2}\right)}{4a^2} \quad (15)$$

Substituting (14), (15) in (13) and employing the method of factorization, define

$$X + i\sqrt{4a^2-b^2}\,T = \left(p+i\sqrt{4a^2-b^2}\,q\right)^2 (4a^2)\frac{\left(b+i\sqrt{4a^2-b^2}\right)}{2a}$$

Equating real and imaginary parts, we've

$X = 2a\,[p^2 b - (4a^2 - b^2)(bq^2 + 2pq)]$
$T = 2a\,[p^2 + 2bpq - (4a^2 - b^2)\,q^2]$

Substituting the values of X and T in (4) and using (2), the values of x and y are given by

$$\left.\begin{aligned} x &= x(a, b, p, q) = 4a\,[2bp^2 + 2b^2pq - (4a^2 - b^2)(2bq^2 + 2pq)] \\ y &= y(a, b, p, q) = 8a^2\,[p^2 + 2bpq - (4a^2 - b^2)\,q^2] \end{aligned}\right\} \quad (16)$$

Thus (14) and (16) represent the integer solutions to (1).

Properties

1. $z(1, 1, p, 1) - t_{10,p} \equiv 0 \pmod 3$
2. $x(a, 1, 1, 1) + y(a, 1, 1, 1) + 2z(a, 1, 1, 1) + 128\,P_a^5 - t_{194,a} \equiv 0 \pmod{127}$
3. $y(a, a+1, 1, 1) + 2\,z(a, a+1, 1, 1) + 32\,P_a^5 - t_{34,a} \equiv 0 \pmod{19}$

Method: 5

Assume $z = z(a, b, A, B) = (2a - b)\,A^2 + (2a + b)\,B^2$ (17)

Write 4a as

$$4a = \left(\sqrt{2a-b} + i\sqrt{2a+b}\right)\left(\sqrt{2a-b} - i\sqrt{2a+b}\right) \quad (18)$$

Substituting (17) and (18) in (3) and employing the method of factorization, define

$$\sqrt{2a-b}\;U + i\sqrt{2a+b}\;V = \left(\sqrt{2a-b} + i\sqrt{2a+b}\right)\left(\sqrt{2a-b}\;A + i\sqrt{2a+b}\;B\right)^2$$

Equating real and imaginary parts, we've

$U = (2a - b)\,A^2 - (2a + b)\,B^2 + 2\,(2a + b)\,AB$
$V = (2a - b)\,A^2 - (2a + b)\,B^2 - 2\,(2a - b)\,AB$

Substituting the values of U and V in (2), the values of x and y are given by

$$\left.\begin{aligned} x &= x(a, b, A, B) = 2A^2(2a - b) - 2B^2(2a + b) + 4bAB \\ y &= y(a, b, A, B) = 8aAB \end{aligned}\right\} \quad (19)$$

Thus (17) and (19) represents the integer solutions to (1).

Method: 6

When $(4a^2 - b^2)$ is not a perfect square, (5) may be represented as the system of double equations in the following ways

System 1: $z + X = (2a + b)\,T$
$z - X = (2a - b)\,T$

System 2: $z + X = T^2$
$z - X = 4a^2 - b^2$

System 3: $z + X = (2a + b)\,T^2$
$z - X = (2a - b)$

Solving each of the above system of equations the corresponding integer solutions obtained are represented below.

Solution for system 1

$x = 4bT$
$y = 4aT$
$z = 2aT$

Solution for system 2

For this system, there are two choices of solutions.

Choice 1

$b = 2B,\ T = 2C.$

$$x = 4\left(B^2 + C^2 - a^2\right) + 8BC$$
$$y = 8ac$$
$$z = 2\left(a^2 + C^2 - B^2\right)$$

Choice 2

$b = 2B+1,\ T = 2C+1.$

$$x = 4\left(B^2 + C^2 + B + C - a^2\right) + 2 + 2\left(4BC + 2B + 2C + 1\right)$$
$$y = 4a\left(2C + 1\right)$$
$$z = 2\left(a^2 + C^2 - B^2 + C - B\right)$$

Solution for system 3

$$x = 2bA\left(A + 1\right)$$
$$y = 4aA$$
$$z = 2A^2 a$$

Method: 7

Let $a = 2\,(R^2 + S^2)$, $b = 8RS$, where R and S are non-zero integers such that $R > S > 0$.

Then, $4a^2 - b^2 = \left[4\left(R^2 - S^2\right)\right]^2 = B^2$ say.

In this case, (5) is written as

$$X^2 + B^2\,T^2 = z^2$$

Which is in the form of Pythagorean equation.

Using the most cited solution of the Pythagorean equation, we've

$$\begin{aligned} X &= 2\,B^2\,MN \\ T &= B\,(M^2 - N^2),\ M > N > 0 \\ Z &= B^2\,(M^2 + N^2) \end{aligned} \quad (20)$$

Substituting X and T in (4) and using (2), the values of x and y are given by

$$\begin{aligned} x &= x(R, S, M, N) \\ &= 64MN\,(R^2 - S^2) + 64RS\,(R^2 - S^2)(M^2 - N^2) \\ y &= y(R, S, M, N) \\ &= 32\,(R^4 - S^4)\,(M^2 - N^2) \end{aligned} \quad (21)$$

Thus (20) and (21) represent the integer solutions to (1).

3. Conclusion

In this paper, we have obtained infinetly many non-zero distinct integer solutions to the ternary quadratic Diophantine represented by $a(x^2 + y^2) - bxy = 4az^2, b \neq 2a$.

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