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Study of wave propagation with three phase lag theory

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Abstract
In the present paper, we have introduced the three phase lag theory to study the dynamical interactions in a thermoelastic medium. Normal mode analysis technique is employed onto the non-dimensional field equations to obtain the analytical solution. The numerical estimates of the field variables displacement, stress and temperature are computed for magnesium crystal like material and presented graphically. Comparisons of the physical quantities are shown in figures to study the effect of three phase lag parameters and time. Numerical results predict finite speed of propagation for thermoelastic waves.

Keywords: Three phase lag, Thermoelasticity, Normal mode analysis

1. Introduction

The conventional theory of thermoelasticity can be used in several particular problems, although this turns out to predict an infinite speed of thermal signals, which is physically unrealistic. Generalized theories proposed by Lord and Shulman (1967) [7] and Green and Lindsay (1972) [3] are two well known theories of thermoelasticity to overcome this deficiency. After that, providing sufficient basic modifications in governing equations, Green and Naghdi (1991, 1992, 1993) [4, 5, 6] produced an alternative theory which was further divided into three different parts, referred to as GN theory of type I, II, III. A remarkable generalization of the coupled theory of thermoelasticity is referred to as dual phase lag thermoelasticity proposed by Tzou (1995) [10] and Chandrasekharaiah (1998) [1]. Tzou (1995) [10] introduced two phase lags, one for heat flux vector and the other for temperature gradient. An another generalization, known as three phase lag thermoelasticity, was developed by Roychoudhuri (2007) [9]. In this generalization, the Fourier’s law of heat conduction is replaced by an approximation to a modification of the Fourier’s law with introduction of three different phase lags for the heat flux vector, the temperature gradient and thermal displacement gradient. The stability of the three phase lag heat conduction equation and relation among the three phase lag parameters are discussed in detail by Quintanilla and Racke (2008) [8].

The objective of the present investigation is to study the phenomenon of wave propagation in a generalized thermoelastic medium with three phase lags subjected to thermal load. Normal mode analysis is adopted to solve the governing equations. Numerical results for temperature, displacement and stress distributions have been obtained and presented graphically. Comparisons are made with results predicted by three phase lag model and GN-III theory.

2. Governing equations

Following the thermoelasticity theory with three phase lags Roychoudhuri (2007) [9], the constitutive relations and field equations in the absence of body force for an isotropic, homogeneous, thermoelastic solid can be expressed as follows:

(i) Constitutive relations

$$\sigma_{ij} = \lambda u_{r,r}\delta_{ij} + \mu\left(u_{j,i} + u_{i,j}\right) - \beta_1\theta\delta_{ij}, \tag{1}$$

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2}$$

(ii) Stress equation of motion

$$\sigma_{ji,j} = \rho \ddot{u}_i, \tag{3}$$

(iii) Equation of heat conduction

$$\left[k_1\left(1+\tau_v\frac{\partial}{\partial t}\right)+k^*\frac{\partial}{\partial t}\left(1+\tau_T\frac{\partial}{\partial t}\right)\right]\nabla^2\theta$$

$$=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho C_E\ddot{\theta}+\beta_1 T_0\ddot{e}\right) \tag{4}$$

Here, u_i are components of the displacement vector $\vec{u}$, σ_{ij} are components of the stress tensor, e_{ij} are the components of strain tensor, T is the absolute temperature, T_0 is the reference temperature chosen so that $\left|(T-T_0)/T_0\right| << 1, \vec{\phi}$ is Microrotation vector, θ is temperature deviation from reference temperature i.e. $\theta = T - T_0$, λ, μ are Lame's constants, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, α_t is coefficient of linear thermal expansion, ρ is density of the medium, *e* is cubical dilatation, C_E is specific heat at constant strain, k^* is thermal conductivity, k_1 is material constant characteristic of the theory, τ_T is phase lag for the temperature gradient, τ_q is phase lag for heat flux, τ_v is phase lag for the thermal displacement gradient, a comma followed by suffix denotes material derivative and a superposed dot denotes the derivative with respect to time *t*.

3. Problem formulation

Let us consider a homogeneous, isotropic, generalized thermoelastic medium. The rectangular Cartesian co-ordinates are introduced having origin on the surface (*z* = 0) and *z*-axis pointing vertically downwards into the medium. All the quantities related to the medium considered will be functions of the time variable *t* and the coordinates *x* and *z*. For a two dimensional problem in the *x*-*z* plane, we can write the displacement vector as

$$u = u_x = u(x,z,t),\quad v = u_y = 0,\ \text{and}\ w = u_z = w(x,z,t) \tag{5}$$

Substitution of Eqs. (2) and (5) into Eqs. (1)

$$\sigma_{xx} = (2\mu+\lambda)\frac{\partial u}{\partial x}+\lambda\frac{\partial w}{\partial z}-\beta_1\theta, \tag{6}$$

$$\sigma_{zz} = (2\mu+\lambda)\frac{\partial w}{\partial z}+\lambda\frac{\partial u}{\partial x}-\beta_1\theta, \tag{7}$$

$$\sigma_{xz} = \sigma_{zx} = \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right). \tag{8}$$

With the aid of expressions (1), (2) and (5), the equations of motion (3), take the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \mu\nabla^2 u+(\mu+\lambda)\frac{\partial^2 u}{\partial x^2}+(\mu+\lambda)\frac{\partial^2 w}{\partial z\partial x}-\beta_1\frac{\partial\theta}{\partial x}, \tag{9}$$

$$\rho\frac{\partial^2 w}{\partial t^2} = \mu\nabla^2 w+(\mu+\lambda)\frac{\partial^2 w}{\partial z^2}+(\mu+\lambda)\frac{\partial^2 u}{\partial z\partial x}-\beta_1\frac{\partial\theta}{\partial z}, \tag{10}$$

The governing equations can be put into more convenient forms by introducing the following non-dimensional variables:

$$(x',z') = \frac{w^*}{c_1}(x,z),\quad (u',w') = \frac{\rho w^* c_1}{\beta_1 T_0}(u,w),$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\beta_1 T_0},\quad (t',t'_p,\tau'_T,\tau'_q,\tau'_v) = w^*(t,t_p,\tau_T,\tau_q,\tau_v),$$

$$\theta' = \frac{\theta}{T_0}, \tag{11}$$

where

$$w^* = \frac{\rho C_E c_1^2}{k^*}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}.$$

Using Helmholtz decomposition, the displacement components can be written as

$$u = \frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial z}, \quad w = \frac{\partial q}{\partial z} - \frac{\partial \psi}{\partial x}, \quad \psi = \left(-\vec{U}\right)_y, \tag{12}$$

Where $q(x,z,t)$ and $\psi(x,z,t)$ are scalar potential functions and $\vec{U}(x,z,t)$ is the vector potential function.
Now, in terms of the dimensionless quantities given in (11), Eqs. (9) - (10), (4) and (6)-(8) with the aid of expressions (12) along with some simplifications, assume the forms (after dropping the primes)

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) q - \theta = 0, \tag{13}$$

$$\left(\nabla^2 - a_1 \frac{\partial^2}{\partial t^2}\right) \psi = 0, \tag{14}$$

$$\left[k_1\left(1 + \tau_v \frac{\partial}{\partial t}\right) + k^* \omega^* \frac{\partial}{\partial t}\left(1 + \tau_T \frac{\partial}{\partial t}\right)\right] \nabla^2 \theta$$

$$= \left(1 + \tau_q \frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right) \frac{\partial^2}{\partial t^2}\left(a_2 \theta + a_3 \nabla^2 q\right), \tag{15}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} + a_1 \frac{\partial w}{\partial z} - \theta, \tag{16}$$

$$\sigma_{zz} = \frac{\partial w}{\partial z} + a_4 \frac{\partial u}{\partial x} - \theta, \tag{17}$$

$$\sigma_{xz} = \sigma_{zx} = a_2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \tag{18}$$

where

$$a_1 = \frac{\rho c_1^2}{\mu}, \quad a_2 = \rho c_1^2 C_E, \quad a_3 = \frac{\beta_1^2 T_0}{\rho}, \quad a_4 = \frac{\lambda}{\rho c_1^2}.$$

4. Solution of the problem

Solution of the physical quantities can be decomposed in terms of normal modes in the following form

$$\left(u, w, q, \psi, \theta, \sigma_{ij}\right)(x,z,t) = \left(u^*, w^*, q^*, \psi^*, \theta^*, \sigma_{ij}^*\right)(z)\, e^{(\omega t + \iota m x)}, \tag{19}$$

where $u^*(z), w^*(z), q^*(z), \psi^*(z), \phi_2^*(z), \theta^*(z), \sigma_{ij}^*(z)$ are the amplitudes of the functions, ω is the angular frequency, ι is the imaginary unit and *m* is the wave number in *x* direction.
Using (19) in the Eqs. (13)-(15), we obtain the following differential equations

$$\left(D^2 - G_1\right)\psi^*(z) = 0, \tag{20}$$

$$\left(D^4 - G_2 D^2 + G_3\right)\left\{q^*(z), \theta^*(z)\right\} = 0, \tag{21}$$

where $D = \frac{d}{dz}$, $a_4 = \frac{\lambda}{\lambda + 2\mu}$, $G_1 = m^2 + a_1 w^2$, $G_2 = \frac{b_5}{b_1}$, $G_3 = \frac{b_6}{b_1}$,

$$b_1 = k_1\left(1 + \tau_v \omega\right) + k^* \omega^* \omega\left(1 + \tau_T \omega\right), \quad b_2 = \left(1 + \tau_q \omega + \frac{\tau_q^2 \omega^2}{2}\right)\omega^2 a_2,$$

$$b_3 = \left(1 + \tau_q \omega + \frac{\tau_q^2 \omega^2}{2}\right)\omega^2 a_3, \quad b_4 = m^2 b_1 + b_2, \quad b_5 = b_1 + b_4 + b_2, \quad b_6 = b_4 b_1 + b_2 m^2.$$

Since the intent is that the solutions vanish at infinity so as to satisfy the regularity condition at infinity (which are assumed to be bounded as $z \to \infty$), we can express $\psi^*(x,z,t)$, $q^*(x,z,t)$, $\theta^*(x,z,t)$ in the following forms:

$$\psi(x,z,t) = \left(H_{1i}R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{22}$$

$$q(x,z,t) = \left(\sum_{i=2}^{3} R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{23}$$

$$\theta(x,z,t) = \left(\sum_{i=2}^{3} H_{1i}R_i(m,\omega)e^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{24}$$

where λ_1^2 and λ_2^2, λ_3^2 are roots with positive real parts of the characteristic equations (20) and (21) respectively, $R_i(m,\omega)$, $(i=1,2,3)$ are parameters, depending upon *m* and ω, and

$$H_{11} = -\left(\frac{a_2}{\lambda_i^2 - b_2}\right),\quad H_{1i} = \left(\lambda_i^2 - b_1\right)(i=2,3).$$

Application on normal mode analysis to the expressions for stress components (16), (18) and displacement components (12), in combination with the relations (22)-(24), yields

$$u(x,z,t) = \left(\sum_{i=1}^{3} H_{2i}R_ie^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{24}$$

$$w(x,z,t) = \left(\sum_{i=1}^{3} H_{3i}R_ie^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{26}$$

$$\sigma_{zx}(x,z,t) = \left(\sum_{i=1}^{3} H_{4i}R_ie^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{27}$$

$$\sigma_{zz}(x,z,t) = \left(\sum_{i=1}^{3} H_{5i}R_ie^{-\lambda_i z}\right)e^{(\omega t+\iota m x)}, \tag{28}$$

where $H_{21} = -\lambda_1 H_{11}$, $H_{2i} = \iota m\ (i=2,3)$, $H_{31} = -\iota m H_{11}$

$H_{3i} = -\lambda_i\ (i=2,3)$, $H_{41} = \left(a_4\lambda_1^2 + a_5 m^2\right)H_{11} - a_6$

$H_{4i} = -\left(a_5 + a_6\right)m\iota\lambda_i\ (i=2,3)$, $H_{51} = \left(1-a_5\right)\iota m\lambda_1 H_{11}$

$H_{5i} = \lambda_i^2 - a_4 m^2 - H_{1i}\ (i=2,3)$.

5. Application: Thermal load acting on the surface

Mathematically, boundary conditions of the problem can be expressed as

$$\sigma_{zz}(x,0,t) = 0, \tag{29}$$

$$\sigma_{zx}(x,0,t) = 0, \tag{30}$$

$$\theta(x,0,t) = f(x,t). \tag{31}$$

Applying normal mode analysis to Eqs. (29)-(31) and using Eqs. (24) and (27)-(28) in the resulting equations, we arrive at a non-homogeneous system of linear equations which can be written in the matrix form as

$$\begin{bmatrix} 0 & H_{12} & H_{13} \\ H_{41} & H_{42} & H_{43} \\ H_{51} & H_{52} & H_{53} \end{bmatrix}\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} M_1 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

Solution of system (32) provides us the values of $R_i\ (i=1,2,3)$ as:

$$R_1 = \frac{\Delta_1}{\Delta},\quad R_2 = \frac{\Delta_2}{\Delta},\quad R_3 = \frac{\Delta_3}{\Delta}, \tag{33}$$

where

$\Delta = H_{12}\left(H_{41}H_{53} - H_{43}H_{51}\right) - H_{13}\left(H_{41}H_{52} - H_{42}H_{51}\right)$,

$\Delta_1 = M_1\left(H_{42}H_{53} - H_{43}H_{52}\right)$, $\Delta_2 = M_1\left(H_{41}H_{53} - H_{43}H_{51}\right)$,

$\Delta_3 = M_1\left(H_{41}H_{52} - H_{42}H_{51}\right) M_1 = \left|f^*\right| e^{-(\omega t + \iota m x)}$,

Substitution of (33) into expressions (24) and (28) leads to the expression of field variables as:

$$\theta(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3}\Delta_i H_{1i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)}, \tag{34}$$

$$u(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3}\Delta_i H_{2i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \tag{35}$$

$$w(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3}\Delta_i H_{3i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \tag{36}$$

$$\sigma_{zx}(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3}\Delta_i H_{4i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \tag{37}$$

$$\sigma_{zz}(x,z,t) = \left(\frac{1}{\Delta}\sum_{i=2}^{3}\Delta_i H_{5i} e^{-\lambda_i z}\right) e^{(\omega t + \iota m x)} \tag{38}$$

6. Numerical results and discussions

The dynamical interactions between thermal and mechanical fields in solids have many applications in aeronautics, nuclear reactors and high energy particle accelerators. To understand the interaction phenomena, we have evaluated the numerical results of non-dimensional displacement component *w*, normal stress σ_{zz}, and temperature θ and displayed graphically. For numerical computation, we take the following values of relevant parameters for magnesium crystal like material:

$$\rho = 1.74\times10^{3}\,kg\,m^{-3},\quad k = 1.0\times10^{10}\,kg\,m^{-1}\,s^{-2},$$
$$k^* = 2.510\,W m^{-1}k^{-1},\quad C_E = 9.623\times10^{2}\,J\,kg^{-1}k^{-1},$$
$$\alpha_t = 2.36\times10^{-5}\,K^{-1},\quad T_0 = 293\,K,$$
$$\tau_q = 0.2\,s,\quad \tau_T = 0.15\,s,\quad \tau_v = 0.1\,s. \tag{39}$$

We analyze the effects of phase lag parameters and time *t* on the fields. In (Figs. 1-3), the behaviour of the field variables is analyzed to show the effect of phase lag parameters for two different times (t = 0.01, 0.1). (i) three phase lag generalized thermoelasticity (TPLMIS, solid line), (ii) three phase lag generalized thermoelasticity (TPLM, long-dashed line) and (iii) three phase lag generalized thermoelasticity (TPLIS, small-dashed line).

Figure 1 represents the behaviour of displacement distribution *w* with distance *z* to investigate the effects of phase lags for different values of time *t*. Displacement field starts with negative values, which is due to the fact that as the surface of the half space (z = 0) is exposed to thermal source then it undergoes thermal expansion deformation and moves towards the unconstrained direction. With the passage of time, the expansion part enlarges and moves inside the medium dynamically, causing the negative to positive region transform. In addition, the increase in the value of time *t* results in increase in the numerical values of displacement field. Hence, it has increasing effect. The qualitative behaviour is almost the same for all the cases. Figure 2 describes the behaviour of normal stress distribution with distance *z* under the different cases considered. The normal stress distribution for TPLMIS theory has small values in comparison to GN-III theory which clearly indicates that phase lags have decreasing effect and these values increase with increase of time *t*. Figure also indicates that the effect of time *t* is more pronounced than the effect of phase lags. However, for all the cases, normal stress predicts the same behaviour. The dynamic interactions of the temperature deviation θ are presented in Figure 3. It can be observed that the magnitude of temperature decreases with the increase in distance and finally goes to zero. However, the absence of phase lags causes to push the vibrations in upward direction. Although, the values of temperature field increase with time *t*. Moreover, the phenomena of finite wave speed is also predicted by this field.

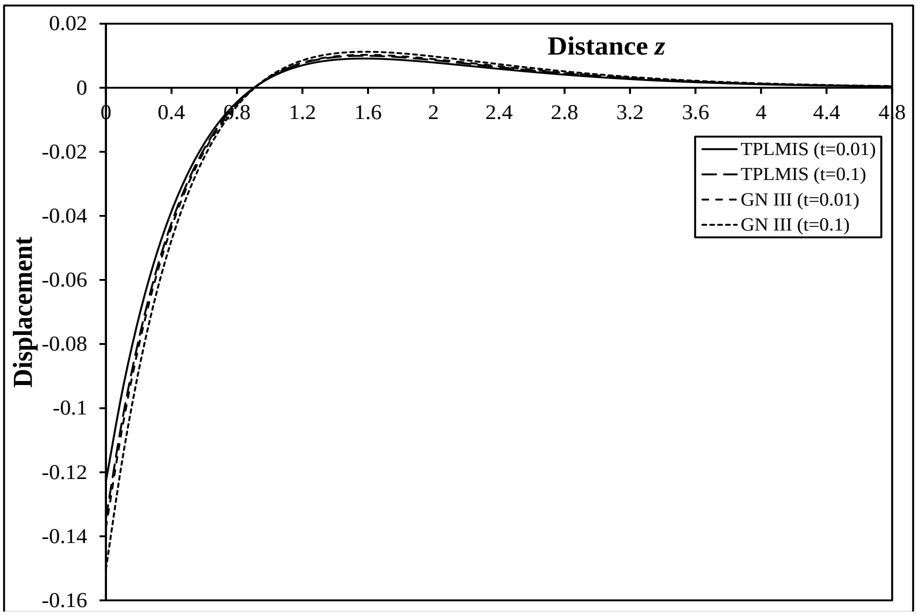

Fig 1

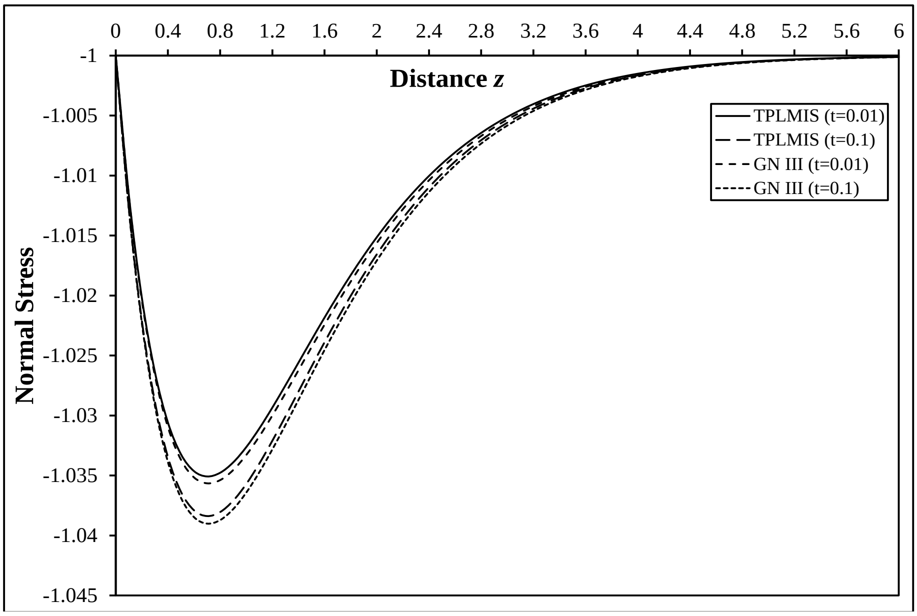

Fig 2

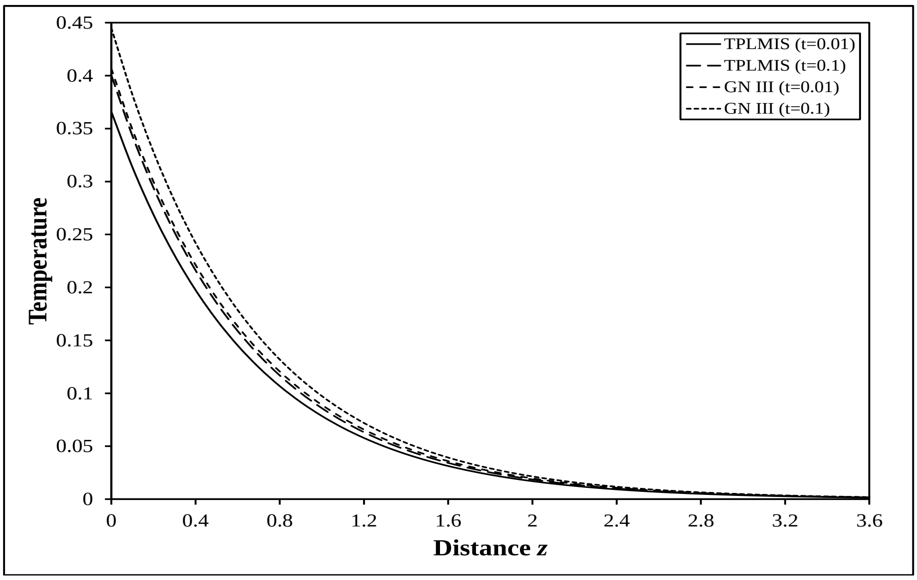

Fig 3

7. Concluding remarks

The present investigation is intended to obtain two dimensional solutions of displacement, temperature thermoelasticity with three phase lags. Normal mode analysis technique is used which has the advantage of finding the exact solutions without any assumed restrictions on the field variables. The numerical work has been carried out with the help of computer programming using the software matlab. Following concluding remarks can be obtained according to the result of present study:

1. All the field variables have non-zero values in a bounded region of space except normal stress. Outside this region, values vanish identically and this means that the region has not felt thermal disturbance yet.
2. All the fields (Figs. 1-3) show significant sensitivity towards the phase lags. The presence of phase lag parameters decreases the magnitude of the field variables.

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