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Parvinder Singh
P.G., Department of Mathematics, S.G.G.S. Khalsa College, Mahilpur, (Punjab).

Correspondence:
Parvinder Singh
P.G. Department of Mathematics, S.G.G.S. Khalsa College, Mahilpur, (Punjab).

Baire Measures and its Unique Extension to a Regular Boral Measure

Parvinder Singh

Abstract
A Baire measure is a probability measure on the Baire $\sigma - algebra$ over a normal Hausdorff space X. A Borel measure is a probability measure on the Borel $\sigma - algebra$ over a normal Hausdorff space X. In this paper we prove that every Baire measure has a unique extension to a regular Borel measure.

Keywords: Baire measures, unique extension**,** Borel measure

1. Introduction
Baire measure is a measure on $\sigma - algebr$ of Baire sets of a topological space X whose value on every compact Baire set is finite. In theory of measure and integration the Baire sets of a locally compact Hausdorff space form a $\sigma - algebra$ related to continuous functions on the space. There are in equivalent definitions of Baire sets which are coincide with case of locally compact and σ-compact Hausdorff spaces. If X be a topological space, Baire sets are those subsets of X belonging to the smallest $\sigma - algebra$ containing all zero sets in X where a zero set is defined as under:

Definition: A set Z $\subset X$ is called a zero set if Z = $f^{-1}_{(0)}$for some continuous real valued function f on X.

Definition: Let X be a topological space then Baire $\sigma - algebra$ $\mathfrak{B}_0$(X) on X is the smallest $\sigma - algebra$ containing the pre-images of all continuous functions f: X→ X. And if there exist a measure μ on $\mathfrak{B}_0$(X) s.t. $\mu(X) < \infty$. Then μ is called a finite Baire measure on X. Further if $\mathfrak{B}(X)$ is the Borel $\sigma - algebra$ on X (i.e. the smallest $\sigma - algebra$ containing the open sets of X) then $\mathfrak{B}_0$(X)$\subset$ $\mathfrak{B}(X)$.

Definition: Borel sets are those sets of X belonging to the smallest $\sigma - algebra$ that contains all closed subsets of X. Clearly a Baire set is always a Borel set. But in many familiar spaces including all metric spaces the classes of all Baire sets and Borel sets are coincide.

Remark: If X be the metric space then $\mathfrak{B}_0$(X) = $\mathfrak{B}(X)$.

Regular Borel Measure: Let μ be a Borel measure on a space X and letE $\in \mathfrak{B}$.We say that the measure μ is outer regular on E if $\mu(E) = inf\{\mu(U): E \subset U, U\ is\ open\}$ and we say that measure μ is inner regular on E if $\mu(E) = sup\{\mu(K) : K \subset U, K\ is\ compact\ \}$. If μ is both inner and outer regular on E then we say that μ is regularon E. Further μ is called Regular Borel measure if it is regular on every Borel set. For example a Radon measure is a Borel measure which is

a. Finite on every compact set.
b. Outer regular on every Borel set.
c. Inner regular on every open set.

Preposition: Let μ be a Borel measure which is finite on compact sets. Then the following statements are equivalent.
1. μ is outer regular on σ-bounded sets.
2. μ is inner regular on σ-bounded sets.
$Proof$: (1) ⇒ (2) Suppose that E is a bounded Borel set and $E \subset L$ Where L is compact. Assume that $\varepsilon > 0$. We have to prove that there is a compact set $K \subseteq E$ with μ (K) ≥ μ (E)-ε. As the relative compliment L/E is bounded, by outer regularity there is an open set O⊇ L/E such that μ (O) ≤ μ (L/E) +ε. It follows that K = L/O = L∩ O^c is a compact set of E satisfying μ(K) = μ(L) - μ(L∩ O) ≥ μ (L)- μ(O) ≥ μ(L/E)-ε, as required.
$In\ general\ let\ E = E_1 \cup E_2 \cup E_3 \cup ....$ is a countable union of bounded Borel sets E_i. We may assume that the sets $E_i$are disjoint. If some of the E_i has finite measure, then by above we have Sup{μ(K): K $\subseteq E_i$, K ∈ $\mathcal{k}$ } $= \mu(E_i) = +\infty$, where $\mathcal{k}$ is the family of compact sets. Then Sup{μ(K): K $\subseteq E$, K ∈ $\mathcal{k}$ } $=\mu(E) = +\infty$ Proved.
But on the other hand if $\mu(E_i) < \infty$ for each i then for any $\varepsilon > 0$ we can find a sequence of compact sets $K_i \subseteq E_i$ with the property that $\mu(E_i) \leq \mu(K_i) + \frac{\varepsilon}{2^i}$.
Taking $L_n= K_1 \cup K_2 \cup ...... \cup K_n$, it is clear that L_n is a compact subset of E for which
$\mu(L_n)=\sum_{i=}^{n}\mu(K_i) \geq \sum_{i=}^{n}\mu(K_i) - \frac{\varepsilon}{2^i} \geq \sum_{i=}^{n}\mu(K_i) - \varepsilon$. Taking supremum over n we get $Sup\,\mu(L_n) \geq \mu(E) - \varepsilon$. Which shows that μ is Inner regular on σ-bounded sets.
(2) ⇒ (1) Let E be abounded Borel set.Then closure of E is $\bar{E}$and is compact set and by single covring there exist a bounded open set U s.t.$\bar{E} \subseteq U$. Let $\varepsilon > 0$ then L/E is a bounded Borel set, then by Inner regularity there exist a bounded compact set K ⊆ L/E $with$ the property that μ(K) ≥ μ(L/E) − ε. Let V = U/K = U∩ K^c then V is a bounded and open which contains E and μ(V)= μ(U ∩ K^c) ≤ μ(L ∩ K^c) = μ(L) - μ(K) ≤ $\left(\mu\left(\frac{L}{E}\right) - \varepsilon\right)$= μ(E) − ε. As ε is arbitrary positive and this proves that μ is outer regular on bounded sets.
Further let E = $\bigcup_n E_n$ where each E_n is a bounded Borel set and each E_i is disjoint and $\mu(E_n) < \infty$ for all n. From the above we have a sequence of open sets $O_n \supseteq E_n$ such that $(O_n) \leq \mu(E_n) + \frac{\varepsilon}{2^n}$. Therefore the set E is contained in the union of O=$\bigcup_n O_n$ and we get $\mu(O) \leq \sum_{i=}^{n}\mu(O_i) \leq \sum_{i=}^{n}\mu(E_i) + \varepsilon = \mu(E) + \varepsilon$.Hence the proof.

Content: A real valued function defined on a $\sigma - algebra\ \mathcal{A}$ of sub sets of a space X is said to be a content on X if 1.) $\mu(A) > 0$ for all A ∈ $\mathcal{A}$.
2.) $\mu(\emptyset) = 0$ and
3.) $\mu(A_1 \cup A_2)= \mu(A_1) + \mu(A_2)$ for all $A_1, A_2 \in \mathcal{A}$.

Preposition: Let λ be content on X and μ be a regular Borel measure induced by λ, then the following statements are equivalent.
1. λ is regular content.
2. μ is an extension of λ.
Proof: Suppose that (1) holds i.e. λ is a regular content. Let C be any compact set, and $\varepsilon > 0$, By the regularity of λ, we can find a compact set D s.t.
C⊂D and $\lambda(D) \leq \lambda$(C) + ε ....(1)
Let λ^* be the outer measure induced by λ. Then let U = D then C⊂U ⊂D and U is an open bounded Borel set, we have
$\lambda^*(C) \leq \lambda^*(U)$ $\{Because\ \lambda^*\ is\ monotone\}$
$\leq \lambda(D)$) ≤ λ(C) + ε $\{By\ (1)\}$
Hence $\lambda^*(C) \leq \lambda$(C) But $\mu(C) = \lambda^*(c)$ Therefore $\lambda^*(C) = \lambda$(C) i.e. $\mu(C) = \lambda$(C) which shows that μ is an extension of λ.
Now suppose that (2) holds, that means μ is an extension of λ. Hence λ is restriction of $\mu\ and\ \mu$ is regular Borel measure. Hence by case (1) λ is a regular content.

Remark: Let C and D be two disjoint sets of X, and then there exist open bounded Baire sets U and V s.t. C ⊂U and D ⊂V.
Proof: Let $x \in C$ then $x \notin D$ we can find disjoint open sets U_x and V_x s.t. .$x \in U_x$, D⊂ V_x . It is clear that $\{V_x / x \in C\}$ is an open cover for C and C is compact, hence there exist a finite sub cover x_1, x_2.........x_n s.t. C ⊂ $\bigcup_{i=1}^{n} U_{x_i}$. Let U^*= $\bigcup_{i=1}^{n} U_{x_i}\ and\ V^* = \bigcap_{i=1}^{n} V_{x_i}$ then $U^*\ and\ V^*$ are disjoint open sets and C ⊂ U^* and D ⊂ V^*, By Baire sandwich theorem there exist open bounded Baire sets U and V s.t. C ⊂U ⊂ U^* and D ⊂V D ⊂ V^* and obviously U and V are disjoint.

Main Result: Every Baire measure has a unique extension to a regular Borel measure.

Lemma: Let υ be any Baire measure on X. Define for compact set C
$\lambda(C) = Inf\{\upsilon(C)/\ C \subset U, U\ is\ open\ Baire\ set\}$. Then λ is a regular content and $\lambda(D) = \mu(D)$ for every compact G_δ set D.

Proof of the Lemma: (1) Since $\nu \geq 0$ then obviously λ ≥0.
Let C be any compact set, By Baire sandwich theorem we can find an open Baire set U and a compact G_δ set D s.t.C ⊂U ⊂ D. Which gives that $\lambda(C) \leq \nu(U) \leq \nu(D) < \infty$ which shows that λ is real valued.
(2) Suppose C and D are two compact sets and C⊂ D. Let U be any open Borel set s.t. D ⊂U then C⊂ $U \Rightarrow \lambda(C) \leq \nu(U) \Rightarrow \lambda(C) = Inf\{\upsilon(U)/\ U\ is\ open\ Baire\ set\ and$ D ⊂ U}.
⇒ $\lambda(C) \leq \lambda(D) \Rightarrow \lambda$ is monotone.
(3) Let C and D be compact sets and U be any open Baire set s.t. C ⊂ U and V any open Baire set s.t. D ⊂ V then C∪D ⊂ U∪V ⇒ λ(C ∪ D) ≤ ν(U ∪ V) ≤ $\nu(U) + \nu(V)$
⇒ λ(C ∪ D) ≤ inf { $\nu(U)$} + inf{$\nu(V)$} ⇒ λ(C ∪ D) ≤ $\lambda(C) + \lambda(D) \Rightarrow \lambda$ is sub additive.
(4) Let C and D be any two disjoint compact sets then by above remark we can find disjoint open bounded Baire sets U and V s.t. C ⊂U and D ⊂V. Let W be an open Baire set s.t. C∪D ⊂W then C ⊂ U∩ W and D ⊂ V∩ W.
$Since\ W$ ⊃(U∩ W) ∪ (V∩ W), we get $\nu(W) \geq \nu$[(U∩ W) ∪ (V∩ W)]=ν(U∩ W)+ ν(V∩ W) ≥ $\lambda(C) + \lambda(D)$.
⇒Inf{$\nu(W)$} ≥ $\lambda(C) + \lambda(D)$ then from (3) we get(C ∪ D) ≤ $\lambda(C) + \lambda(D)$. Hence we get(C ∪ D) = $\lambda(C) + \lambda(D)$. This proves that λ is content.

(5)**Regularity**: Let C be any compact set, $\varepsilon > 0$ then by definition of λ we can find an open Baire set U such that C⊂ U and$(U) \leq \lambda(C) + \varepsilon$. By Baire sandwich theorem we can find an open Baire set V and a compact G_δ set D s.t. C⊂ $V \subset$ D ⊂ U. Then C≼ D and $\lambda(D) \leq \nu(U) \leq \lambda(C) + \varepsilon$. ⇒ $\lambda(C) = Inf\{\lambda(D)/$C ≼ D, D is compact}. This proves that λ is regular content.

(6)**Finally:** Let D be any compact G_δ set. Let (U_n) be any

sequence of open sets s.t. D = $\bigcap_{i=1}^{n}(U_n)$ for each n, D $\subset U_n$,
By Baire sandwich theorem we can find an open Baire set V_n and compact G_δ set D_n s.t. $D \subset V_n \subset D_n \subset U_n \Rightarrow$ D=$\bigcap_{i=1}^{n}(V_n)$.
$Define$ W_n =$\bigcap_{i=1}^{n}(V_i)$. Then (W_n) is a monotone decreasing sequence of open bounded Baire sets s.t. $(W_n) \rightarrow D \Rightarrow \nu(W_n)) \rightarrow \nu(D)$. as $D \subset W_n$ for all n $\Rightarrow \lambda(D) \leq \lim_{n\to\infty} \nu(W_n) \Rightarrow \lambda(D) \leq \nu(D)$..............(*)
$If\ V\ be\ any\ Open\ Baire\ set\ s.t.$ D$\subset$ V then $\nu(D) \leq \nu(V) \Rightarrow \nu(D) \leq \text{Inf}\{\nu(W)\} \Rightarrow \nu(D) \leq \lambda(D)$................(**)
$From\ (*) and (**)$ we get $\lambda(D) = \nu(D)$.

Proof of the Main Theorem: Let ν be any Baire measure on X.
Define for compact set C, $\lambda(C)$=$Inf\{\nu(U)/C \subset U$, U is Baire set$\}$. Then λ is a regular content. Let μ be the regular Borel measure induced by λ. Then Let μ is an extension of λ.Let ν' be the Baire restriction of μ. Let D be any compact G_δ set,
then $\nu(D) = \lambda(D) = \nu'(D)$ [By above lemma]
$\Rightarrow \nu(E) = \nu'(E)$ For all Baire sets E. $\Rightarrow \nu(E) = \mu(E)$ For every Baire set E.
i.e. μ is an extension of ν. Thus Baire measure ν has been extended to a regular Borel measure μ.

Uniqueness: Let μ_1 and μ_2 be two regular Borel measures such that $\mu_1(D) = \mu_2(D)$ for every compact G_δ set D.
To prove above we have to show that $\mu_1(E) = \mu_2(E)$ for every Borel set E. For this it is suffices to prove that $\mu_1(C) = \mu_2(C)$ for every compact set C.
Let C be any compact set. Since $\mu_1$is regular Borel measure we can find a compact G_δ set D_1 such that C $\subset D_1$ and $\mu_1(C) = \mu_1(D_1)$ ....(1)
By the same argument we can find a compact G_δ set D_2 such that C $\subset D_2$ and $\mu_2(C) = \mu_2(D_2)$.......(2)
Define D=$D_1 \cap D_2$, then D is a compact G_δ set and C $\subset D_1$ and C $\subset D_2 \Rightarrow C \subset D$ shows that $\mu_1(C) = \mu_1(D_1) \leq \mu_1(D) = \mu_2(D) \leq \mu_2(D_2) = \mu_2(C) \Rightarrow \mu_1(C)) \leq \mu_2(C)$
By the same argument $\mu_2(C) \leq \mu_1(C)$.
Hence proved that $\mu_1(C) = \mu_2(C)$. Hence the theorem.

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