ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2015; 1(8): 85-90
www.allresearchjournal.com
Received: 15-05-2015
Accepted: 16-06-2015

Shalini Sharma
Assistant Professor,
Department of Mathematics,
Govt. College Faridabad,
Haryana, India

Correspondence
Shalini Sharma
Assistant Professor,
Department of Mathematics,
Govt. College Faridabad,
Haryana, India

Few results of common fixed point theorem for weakly compatible mappings in complex valued metric spaces

Shalini Sharma

Abstract
Azam *et al.* [11], introduce the notion of complex valued metric spaces and obtained common fixed point result for mappings in the context of complex valued metric spaces. In this paper, we prove a common fixed point theorem for weakly compatible mappings in complex valued metric spaces taking an identity function. Our results generalize some recent result in the literature due to Azam [1] and Sintunavarat [10]. We also present an example to support our result.

Keywords: Complex valued metric space, point of coincidence, Cauchy and convergent sequence, weakly compatible mappings, common fixed point

1. Introduction

The concept of complex valued metric pace which is a generalization of the classical metric space was recently introduced by Azam, Fisher and khan [1]. The study of metric space expressed the most common important role to many fields both in pure and applied science [12]. Banach's fixed point theorem plays a major role in fixed point theory. Banach contraction principle was the starting point for many researchers during last few decades in the field of nonlinear analysis. It has application in many branches of Mathematics. Because of its usefulness, a lot of article has been dedicated to the improvement and generalization of that result. Most of these generalizations have been made by considering different contractive type conditions in different spaces. In 2011, Azam [1] made one such generalization by introducing a complex valued metric space. Very recently, Sintunavarat [10] generalized this result by replacing the constant of contraction by some control functions. The purpose of this work is to obtain a common fixed point result for weakly compatible mappings in complex valued metric spaces which generalizes the result of [1] and [10].

2. Perliminaries

Let $\mathbb{C}$ be the set of complex number and let $z_1, z_2 \in \mathbb{C}$. We can define a partial ordering $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ iff Re $(z_1) \leq$ Re (z_2) and Im $(z_1) \leq$ Im (z_2).

It follows that, $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2);
(ii) Re $(z_1) <$ Re (z_2) and Im (z_1) = Im (z_2);
(iii) Re (z_1) = Re (z_2) and Im $(z_1) <$ Im (z_2);
(iv) Re $(z_1) <$ Re (z_2) and Im $(z_1) <$ Im (z_2).

In particular, we will write $z_1 \lneqq z_2$ if $z_1 \neq z_2$ and one of (ii), (iii), and (iv) is satisfied and we will write $z_1 \prec z_2$ if only (iv) is satisfied. Note that

(i) $0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$;
(ii) $0 \precsim z_1 \lneqq z_2 \Rightarrow |z_1| < |z_2|$;
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$;
(iv) A, b∈ R, $0 \leq a \leq b$ and $z_1 \precsim z_2 \Rightarrow az_1 \precsim bz_2$.

Definition 2.1 [1] Let X be a nonempty set. Suppose that the mapping d: X x $X \to \mathbb{C}$ satisfies the following conditions:

1. $\preceq d(x,y)$, for all $x, y \in X$ and $d(x,y)=0$ iff $x = y$;
2. $d(x,y)=d(y,x)$, for all $x, y \in X$
3. $d(x,y) \preceq d(x,z)+d(z,y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space. Note that $d(x,y) \preceq 1+d(x,y)$ and so, $|\frac{d(x,y)}{1+d(x,y)}| \leq 1$.

Example 2.2. [1] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1,z_2)=e^{ik}|z_1 - z_2|$, where $k \in R$. Then (X,d) is a complex valued metric space.

Definition 2.3. [1] Let (X,d) be a complex valued metric space, (x_n) be a sequence in X and $x \in X$.

(i) If for every c ϵ ℂ, with $0 \prec$ c there is $n_0 \in N$ such that for all $n > n_0$, $d(x,y) \prec$ c, then (x_n) is said to be convergent, (x_n) converges to x and x is the limit point of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) If for every c ϵ ℂ, with $0 \prec$ c there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec$ c, where $m \in N$, then (x_n) is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued metric space.

Lemma 2.4. [1] Let (X,d) be a complex valued metric space and let (x_n) be a sequence in X . Then (x_n) converges to x iff $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.5. [1] Let (X,d) be complex valued metric space and let (x_n) be a sequence in X . Then (x_n) is a Cauchy sequence iff $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in N$.

Definition 2.6. [4] Let T and S be self-mappings of a set X . If $w = Tx = Sx$ for some x in X , Then x is called a coincidence point of T and S and w is called a coincidence of T and S .

Definition 2.7. [7] Let T and S be self-mapping of a non-empty set X . The mapping T and S are weakly compatible if $TSx = STx$ whenever $Tx = Sx$.

Definition 2.8. Let (X,d) be a complex valued metric space. A mapping $T : X \to X$ is said to be contractive if $d(Tx,Ty) \preceq d(x,y)$ for all $x, y \in X$.

Definition 2.9. Let (X,d) be a complex valued metric space. A mapping $T : X \to X$ is said to be expensive if there is a real constant c > 1 satisfying $cd(x,y) \preceq d(Tx,Ty)$ for all $x, y \in X$.

Main Results

In this section, we always suppose that $\mathbb{C}$ is the set of complex numbers and $\preceq$ is a partial ordering on $\mathbb{C}$. Throughout the paper we denote by N the set of natural numbers.

Lemma 3.1. [2] Let X be a non-empty set and the mappings S , T :, $I : X \to X$ have a unique point of coincidence v in X . If (S,I) and (T,I) are weakly compatible, then S , T and I have a unique common fixed point.

Theorem 3.2: Let (X,d) be a complex valued metric space and let f , $T : X \to X$ satisfy

$$d(Tx,Ty) \preceq \lambda d(fx, fy) + \frac{\mu d(fx,Ty) d(fy,Ty)}{1+d(fx, fy)}$$

For all $x, y \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. If $T(X) \subseteq f(X)$ and $f(X)$ is complete, then f and T have a unique point of coincidence. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point in X. Choose a point $x_1 \in X$ such that $fx_1 = Tx_0$ which is possible since $T(X) \subseteq f(X)$. Also, we may choose a point $x_2 \in X$ satisfying $fx_2 = Tx_1$ since $T(X) \subseteq f(X)$. Continuing in this way, we can construct a sequence (fx_n) in $f(X)$ such that

$fx_0 = Tx_{n-1}$. if n is odd

$= Tx_{n-1}$. if n is even.

If $n \in N$ is odd, then by using hypothesis we obtain

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \lambda d(fx_{n-1}, fx_n) + \frac{\mu d(fx_{n-1}, Tx_{n-1}) d(fx_n, Tx_n)}{1 + d(fx_{n-1}, fx_n)}$$

$$= \lambda d(fx_{n-1}, fx_n) + \frac{\mu d(fx_{n-1}, fx_n) d(fx_n, Tx_{n+1})}{1 + d(fx_{n-1}, fx_n)}$$

Therefore,

$$|d(fx_n, fx_{n+1})| \leq \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})| \left| \frac{d(fx_n, fx_{n+1})}{1 + d(fx_{n-1}, fx_n)} \right|$$

$$\leq \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$= \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$\leq \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$= \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$\leq \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$\leq \lambda |d(fx_{n-1}, fx_n)| + \mu |d(fx_n, fx_{n+1})|$$

$$|d(fx_n, fx_{n+1})| \leq \frac{\lambda}{1-\mu} |d(fx_{n-1}, fx_n)|$$

Which implies that

If $n \in N$ is even, then

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n) = d(Tx_n, Tx_{n-1})$$

$$\leq \lambda d(fx_n, fx_{n-1}) + \frac{\mu d(fx_n, Tx_n) d(fx_{n-1}, Tx_{n-1})}{1 + d(fx_n, fx_{n-1})}$$

$$= \lambda d(fx_n, fx_{n-1}) + \frac{\mu d(fx_n, fx_{n+1}) d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})}$$

Therefore,

$$|d(fx_n, fx_{n+1})| \le \lambda |d(fx_n, fx_{n-1})| + \mu |d(fx_n, fx_{n+1})| \left| \frac{d(fx_{n-1}, fx_n)}{1 + d(fx_n, fx_{n-1})} \right|$$

$$\le \lambda(fx_n) d(fx_n, fx_{n-1}) + \mu d(fx_n, fx_{n+1})$$

$$= \lambda d(fx_n, fx_{n-1}) + \mu d(fx_n, fx_{n+1})$$

$$\le \lambda |d(fx_n, fx_{n-1})| + \mu |d(fx_n, fx_{n+1})|$$

$$= \lambda |d(fx_n, fx_{n-1})| + \mu |d(fx_n, fx_{n+1})|$$

$$\le \lambda |d(fx_n, fx_{n-1})| + \mu |d(fx_n, fx_{n+1})|$$

Which gives that

$$|d(fx_n, fx_{n+1})| \le \frac{\lambda}{1-\mu} |d(fx_n, fx_{n-1})|$$

Thus for any positive integer n. It must be the case that

$$|d(fx_n, fx_{n+1})| \le \frac{\lambda}{1-\mu} |d(fx_{n-1}, fx_n)| \tag{3.1}$$

If we let $\alpha = \frac{\lambda}{1-\mu}$. Then by repeated application of (3.1)

$$|d(fx_n, fx_{n+1})| \le \alpha |d(fx_{n-1}, fx_n)|$$

$$\le \alpha^2 |d(fx_{n-2}, fx_{n-1})|$$

$$\le \alpha^n |d(fx_0, fx_1)|$$

Now, for all m, $n \in N$, $m > n$, we have

$$d(fx_n, fx_m) \le d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \ldots\ldots\ldots\ldots\ldots + d(fx_{m-1}, fx_m).$$

Therefore,

$$d(fx_n, fx_m) \le |d(fx_n, fx_{n+1})| + |d(fx_{n+1}, fx_{n+2})| + \ldots\ldots\ldots + |d(fx_{m-1}, fx_m)|$$

$$\le (\alpha^n + \alpha^{n+1} + \ldots\ldots\ldots + \alpha^{m-1}) |d(fx_{m-1}, fx_m)|$$

$$\le \frac{\alpha^n}{1-\alpha} |d(fx_0, fx_1)|$$

Since $\alpha \in [0,1)$ taking limit as $m, n \to \infty$ we have $|d(fx_n, fx_m)| \to 0$ which implies that (fx_n) is a Cauchy sequence in $f(X)$. By completeness of $f(X)$, there exist u, $v \in X$ such that

$$fx_n \to v = fu$$

Now,

$$d(fu,Tu) \leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, Tu)$$

$$= d(fu, fx_{2n+1}) + d(Tx_{2n}, Tu)$$

$$\leq d(fu, fx_{2n+1}) + \lambda d(fx_{2n}, fu) + \frac{\mu d(fx_{2n}, Tx_{2n}) d(fu,Tu)}{1 + d(fx_{2n}, fu)}$$

Which implies that

$$|d(fu,Tu)| \leq d(fu, fx_{2n+1}) + \lambda d(fx_{2n}, fu) + \frac{\mu d(fx_{2n}, Tx_{2n}) d(fu,Tu)}{1 + d(fx_{2n}, fu)}$$

$$\leq |d(fu, fx_{2n+1})| + \lambda |d(fx_{2n}, fu)| + \mu |d(fx_{2n}, Tx_{2n})| |d(fu,Tu)|,$$

Since 1≤1+ $d(fx_{2n}, fu)$

$$\leq |d(fu, fx_{2n+1})| + \lambda |d(fx_{2n}, fu)| + \mu |d(fx_{2n}, fx_{2n+1})| |d(fu,Tu)|$$

Taking $n \to \infty$, it follows that $|d(fu,Tu)| = 0$ and hence $d(fu,Tu) = 0$.

Therefore, $fu = Tu = v$. Similarly, we can show that $fu = Tu = v$.

Thus, $fu = Tu = v$ and so v becomes a common point of coincidence of f and T.

For uniqueness, let there exists another point $w(\neq v) \in X$ such that $fx = Tx = w$ for some $x \in X$. Thus,

$$d(v,w) = d(Tu,Tx)$$

$$\leq \lambda d(fu, fx) + \frac{\mu d(fu,Tu) d(fx,Tx)}{1 + d(fu, fx)}$$

$$= \lambda d(v,w) + \frac{\mu d(v,v) d(w,w)}{1 + d(v,w)}$$

$$= \lambda d(v,w)$$

Which implies that

$$|d(v,w)| \leq \lambda |d(v,w)|$$

Since $0 \leq \lambda < 1$, it follows that $|d(v,w)| = 0$ and so $v = w$. If (T, f) is weakly compatible, then by Lemma (3.1), f and T have a unique common fixed point in X.

Example 3.5.Let X= [1, ∞). Define T, f: X → X by Tx = 2x -1 and fx = 5x-4. If d* is the usual metric on X, then T and f are not contraction mappings as for all x, y ϵ X

D*(fx, fy) = 5|x-y|

So, we cannot apply Banach contraction theorem to find the unique fixed point 1 of T and f.

We consider a complex valued metric space d: X x $X \to \mathbb{C}$ by

D (x, y) = |x-y| + i |x-y|

Now,

D (Tx, Ty) = 2 [|x-y| + i|x-y|]
= $\frac{2}{5}$ d (fx, fy)
$\precsim \frac{1}{2}$ d (fx, fy)
Since T(X) = f(X) = X, we have all the onditions of collary 3.4 with λ = $\frac{1}{2}.\, \mu = 0$. So applying

References

1. A Azam, F Brian, M Khan. Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 2011; 32:243-253.
2. A Azam, M Arshad, I Beg. Common fixed point theorems in cone metric spaces, The Journal of Nonlinear Sciences and Applications. 2009; 2:204-213.
3. CT Aage, JN Salunke, Some fixed point theorems for expansion onto mappings on cone metric spaces, Acta Mathematica Sinica English Series. 2011; 27:1101-1106.
4. M Abbas, G Jungck. Common fixed point results for non-commuting mapping without continuity in cone metric spaces, J. Math. Anal. Appl. 2008; 341:416-420.
5. C Di Bari, P Vetro, ϕ-Pairs and common fixed points in cone metric spaces, Rendiconti del Circolo Matematico di Palermo, 2008; 57:279-285.
6. LG Huang, X Zhang, Cone metric Spaces and fixed point theorems of contactive mappings, J. Math. Anal. Appl. 2007; 332:1468-1476.
7. G Jungck. Compatible mapping and common fixed points, International Journal of Mathematics and Mathematical Sciences. 1986; 9:771-779.
8. G Jungck. Common fixed points of commuting and compatible maps on compacta, Proc. Am. Math. Soc, 1988; 103:977-983.
9. G Jungck, Common Fixed points for non-continuous nonself-mapping on a non-numeric spaces, Far East J. Math. Sci. 1996; 4:199-212.
10. W Sintunavarat, P Kumam. Generalized common fixed point theorems in complex valued metric spaces with applications, Journal of Inequalities and Application, doi:10.1186/1029-242X-2012-84.
11. A Azam, B Fisher, M Khan. Common fixed point theorems in Complex valued metric spaces. Numerical Functional Analysis and Optimization. 2011; 32(3):243-253.
12. C Semple, M Steel. Phylogenetics, Oxford Lecture Ser. In Math Appl, vol. 24, Oxford Univ. Press, Oxford, 2003.