ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2015; 1(8): 345-350
www.allresearchjournal.com
Received: 16-05-2015
Accepted: 27-06-2015

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On $b^{\#}$-Separation Axioms

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Abstract
Andrijivic introduced and studied the concept of b-open sets. Following this Usha Parameswari *et al.* introduced the concept of $b^{\#}$-open sets. In this paper, we introduce new types of separation axioms via $b^{\#}$-open sets namely $b^{\#}T_i$-axioms, i= 0, 1, 2, $b^{\#}$-regular spaces, $b^{\#}$-normal spaces and investigate their fundamental properties.

Keywords: b-open sets**,** $b^{\#}$-open sets, $b^{\#}T_i$-axioms, i= 0, 1, 2, $b^{\#}$-regular spaces, $b^{\#}$-normal spaces.
AMS Subject Classification Nos. 2000: 54A05, 54A10.

1. Introduction

Separation axioms in topology are defined to classify the topological spaces and to study certain mathematical objects in analysis. For example the Hausdorff axiom (T_2- axiom) plays a dominant role in the study of analytical structure in mathematics. Researchers in topology have introduced some strong and weak forms of separation axioms by replacing open sets by nearly open sets/generalized open sets and closed sets by nearly closed sets/generalized closed sets.

In this paper $b^{\#}T_i$-axioms, i= 0, 1, 2, $b^{\#}$-regular spaces and $b^{\#}$-normal spaces are introduced and studied.

2. Preliminaries

Throughout this paper, (X, τ), (Y, τ) and (Z, τ) (or simply, X, Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of X, then Cl (A) and Int (A) denote the closure and the interior of A respectivel**y.** The following known definitions and results are used in this paper.

Definition 2.1 [1]: A subset A of a space X is called b-open if $A \subseteq cl(int(A)) \cup int(cl(A))$ and b-closed if $cl(int(A)) \cap int(cl(A)) \subseteq A$,

Definition 2.2: A topological space (X, τ) is said to be

(i) clopen T_1 [2] if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y respectively such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

(ii) ultra Hausdorff [5] if every two distinct points of X can be separated by disjoint clopen sets.

(iii) ultra normal [5] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

(iv) clopen regular [2] if for each clopen set F of X and each $x \notin F$, there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

(v) clopen normal [2] if for each pair of disjoint clopen sets U and V of X there exist two disjoint open sets G and H such that $U \subseteq G$ and $V \subseteq H$.

(vi) ultra regular [3] if for each closed set F of X and each $x \notin F$, there exist disjoint clopen sets U and V such that $F \subseteq U$ and $x \in V$.

Definition 2.3 [4]: A function $f : X \to Y$ is called totally continuous if the inverse image of every open subset of Y is a clopen subset of X.

Definition 2.4 [6]**:** A subset A of a space X is called $b^{\#}$-open if A= $cl(int(A)) \cup int(cl(A))$ and their complement is called $b^{\#}$-closed. That is A is $b^{\#}$-closed if A = $cl(int(A)) \cap int(cl(A))$.

Definition 2.5 [6]**:** Let X and Y are topological spaces. A function f: X $\rightarrow$ Y is called

(i) $b^{\#}$-continuous if $f^{-1}(V)$ is $b^{\#}$-open in X for each open sub set V of Y.
(ii) $b^{\#}$-irresolute if $f^{-1}(V)$ is $b^{\#}$-open in X for each $b^{\#}$-open sub set V of Y.

Lemma 2.6 [6]**.** Let (X, τ) and (Y, σ) be two topological spaces and f be a map from X to Y. Then

(i) f is a $b^{\#}$-continuous map if and only if the inverse image of a closed set in Y is a $b^{\#}$-closed set in X.
(ii) f is a $b^{\#}$- irresolute map if and only if the inverse image of a $b^{\#}$-closed set in Y is a $b^{\#}$-closed set in X.

3. $b^{\#}T_i$-Spaces where i= 0, 1, 2.

In this section we introduce the concepts of $b^{\#}T_i$-Spaces where i= 0, 1, 2 and discuss their properties.

Definition 3.1. Let (X, τ) be a topological space. Then X is said to be

(i) $b^{\#}T_0$ if for any two distinct points x and y of X there exists a $b^{\#}$-open set G such that ($x \in G$ and $y \notin G$) or ($y \in G$ and $x \notin G$).
(ii) $b^{\#}T_1$ if for any two distinct points x and y of X there exist $b^{\#}$-open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.
(iii) $b^{\#}T_2$ if for any two distinct points x and y of X, there exists disjoint $b^{\#}$-open sets G and H such that $x \in G$ and $y \in H$.

Proposition 3.2. If a topological space (X, τ) is a $b^{\#}T_0$-space then any two distinct points of X have disjoint closures.

Proof. Let x and y be any two distinct points of X. Since the space is $b^{\#}T_0$, by Definition 3.1(i), there exists an $b^{\#}$-open set G containing one of them say x, but not containing y. Then X\G is a $b^{\#}$-closed set which does not contain x but contain y. It follows that $x \notin b^{\#}\text{-}cl(y)$ that implies $b^{\#}\text{-}cl(x) \neq b^{\#}\text{-}cl(y)$. Thus in a $b^{\#}T_0$-space distinct points have disjoint $b^{\#}$-closures. We note that $b^{\#}T_2 \Rightarrow b^{\#}T_1 \Rightarrow b^{\#}T_0$.

Theorem 3.3. Let f: (X, τ) $\rightarrow$ (Y, σ) be bijective and $b^{\#}$-continuous.

(i) If (Y, σ) is T_1, then (X, τ) is $b^{\#}T_1$.
(ii) If (Y, σ) is T_2, then (X, τ) is $b^{\#}T_2$.

Proof. Suppose f: (X, τ) $\rightarrow$ (Y, σ) is $b^{\#}$-continuous and (Y, σ) is T_1. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, $y_1=f(x_1) \neq f(x_2)= y_2$ for some $y_1, y_2 \in Y$. Since (Y, σ) is T_1, choose open sets G and H such that $y_1 \in G$ but $y_2 \notin G$ and $y_2 \in H$ but $y_1 \notin H$. Since f is surjective $x_1= f^{-1}(y_1) \in f^{-1}(G)$ and $x_2=f^{-1}(y_2) \notin f^{-1}(G)$ and $x_2= f^{-1}(y_2) \in f^{-1}(H)$ and $x_1= f^{-1}(y_1) \notin f^{-1}(H)$. Since f is $b^{\#}$-continuous, by Definition 2.5(i), $f^{-1}(G)$ and $f^{-1}(H)$ are $b^{\#}$-open sets in (X, τ). This shows that (X, τ) is $b^{\#}T_1$. This proves (i).

Suppose f: (X, τ) $\rightarrow$ (Y, σ) is $b^{\#}$-continuous and (Y, σ) is T_2. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, $y_1=f(x_1) \neq f(x_2)= y_2$ for some $y_1, y_2 \in Y$. Since (Y, σ) is T_2, there exist disjoint open sets G and H such that $y_1 \in G$ and $y_2 \in H$. Since f is surjective $x_1= f^{-1}(y_1) \in f^{-1}(G)$ and $x_2= f^{-1}(y_2) \in f^{-1}(H)$. Since f is $b^{\#}$-continuous, by Definition 2.5(i), $f^{-1}(G)$ and $f^{-1}(H)$ are $b^{\#}$-open sets in (X, τ). Also f is bijective and $G \cap H = \phi$ which implies that $f^{-1}(G \cap H)= f^{-1}(\phi)=\phi$. This implies that $f^{-1}(G) \cap f^{-1}(H)=\phi$. By Definition 3.1(iii), (X, τ) is $b^{\#}T_2$. This proves (ii).

Theorem 3.4. Let f: (X, τ) $\rightarrow$ (Y, σ) be bijective and $b^{\#}$-irresolute.

(i) If (Y, σ) is $b^{\#}T_1$, then (X, τ) is $b^{\#}T_1$.
(ii) If (Y, σ) is $b^{\#}T_2$, then (X, τ) is $b^{\#}T_2$.

Proof. Suppose f: (X, τ) $\rightarrow$ (Y, σ) is $b^{\#}$-irresolute and (Y, σ) is $b^{\#}T_1$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, $y_1=f(x_1) \neq f(x_2)=y_2$ for some $y_1, y_2 \in Y$. Since (Y,σ) is $b^{\#}T_1$, by Definition 3.1(ii), choose $b^{\#}$-open sets G and H such that $y_1 \in G$ but $y_2 \notin G$ and $y_2 \in H$ but $y_1 \notin H$. Since f is surjective, $x_1=f^{-1}(y_1) \in f^{-1}(G)$ and $x_2=f^{-1}(y_2) \notin f^{-1}(G)$ and $x_2=f^{-1}(y_2) \in f^{-1}(H)$ and $x_1=f^{-1}(y_1) \notin f^{-1}(H)$. Since f is $b^{\#}$-irresolute, by Definition 2.5(ii), $f^{-1}(G)$ and $f^{-1}(H)$ are $b^{\#}$-open sets in (X, τ). By Definition 3.1(ii), (X, τ) is $b^{\#}T_1$. This proves (i).

Suppose f: (X, τ) $\rightarrow$ (Y, σ) is $b^{\#}$-irresolute and (Y, σ) is $b^{\#}T_2$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, $y_1=f(x_1) \neq f(x_2)= y_2$ for some $y_1, y_2 \in Y$. Since (Y, σ) is $b^{\#}T_2$, by Definition 3.1(iii), there exist disjoint $b^{\#}$-open sets G and H such that $y_1 \in G$ and $y_2 \in H$. Since f is surjective, $x_1= f^{-1}(y_1) \in f^{-1}(G)$ and $x_2=f^{-1}(y_2) \in f^{-1}(H)$. Since f is $b^{\#}$-irresolute by Definition 2.5(ii), $f^{-1}(G)$ and $f^{-1}(H)$ are $b^{\#}$-open sets in (X, τ). Also f is bijective and $G \cap H=\phi$ which implies that $f^{-1}(G \cap H)= f^{-1}(\phi)=\phi$. This implies that $f^{-1}(G) \cap f^{-1}(H)=\phi$. By Definition 3.1(iii), (X, τ) is $b^{\#}T_2$. This proves (ii).

Definition 3.5: A function f : (X, τ) → (Y, σ) is said to be a $b^{\#}$-totally continuous function if the inverse image of every $b^{\#}$-open subset of Y is clopen in X.

Theorem 3.6. A function f : (X, τ) → (Y, σ) is $b^{\#}$-totally continuous if and only if the inverse image of every $b^{\#}$-closed subset of Y is clopen in X.

Proof. Suppose f is $b^{\#}$-totally continuous. Let B be any $b^{\#}$-closed set in Y. Then Y\B is $b^{\#}$-open set in Y. By Definition 3.5, $f^{-1}(Y \setminus B)$ is clopen in *X*. That is $X \setminus f^{-1}(B)$ is clopen in X. This implies $f^{-1}(B)$ is clopen in X. Conversely, if B is $b^{\#}$-open in Y, then Y\B is $b^{\#}$-closed in Y. By hypothesis, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$ is clopen in X, which implies $f^{-1}(B)$ is clopen in X. Thus, the inverse image of every $b^{\#}$-closed set in Y is clopen in X. Therefore f is $b^{\#}$-totally continuous function.

Definition 3.7: A function f: (X, τ) → (Y, σ) is said to be a totally $b^{\#}$-continuous function if the inverse image of every open subset of Y is a $b^{\#}$-clopen set in X.

Theorem 3.8. A function f : (X, τ) → (Y, σ) is a totally $b^\#$-continuous function if and only if the inverse image of every closed subset of Y is $b^\#$-clopen in X.

Proof. Suppose f is totally $b^\#$-continuous. Let B be any closed set in Y. Then Y\B is open set in Y. By Definition 3.7, f^{-1}(Y\B) is $b^\#$-clopen in X. That is X\$f^{-1}$(B) is $b^\#$-clopen in X. This implies f^{-1}(B) is $b^\#$-clopen in X. Conversely, suppose B is open in Y. Then Y\B is closed in Y. By hypothesis, f^{-1}(Y\B) = X\$f^{-1}$(B) is $b^\#$-clopen in X, which implies f^{-1}(B) is $b^\#$-clopen in X. Thus inverse image of every closed set in Y is $b^\#$-clopen in X. Therefore f is a totally $b^\#$-continuous function.

Theorem 3.9. If f : (X,τ)→(Y, σ) is $b^\#$-totally continuous injection and Y is $b^\#T_1$, then X is clopen T_1.

Proof. Let x and y be any two distinct points in X. Since f is injective we have f(x) and f(y)∈Y such that f(x) ≠ f(y). Since Y is $b^\#T_1$, by Definition 3.1(ii), there exist $b^\#$-open sets U and V in Y such that f(x) ∈ U, f(y) ∉ U, f(y) ∈ V and f(x) ∉ V. Therefore we have x ∈ f^{-1}(U), y ∉ f^{-1}(U), y ∈ f^{-1}(V) and x ∉ f^{-1}(V). Since f is $b^\#$-totally continuous, by Definition 3.5, f^{-1}(U) and f^{-1}(V) are clopen subsets of X. Therefore by Definition 2.2(i) implies that X is clopen T_1.

Theorem 3.10. Let f: (X, τ) → (Y, σ) be injective and totally $b^\#$-continuous.

(i) If (Y, σ) is T_0, then (X, τ) is $b^\#T_2$.
(ii) If (Y, σ) is T_2, then (X, τ) is $b^\#T_2$.

Proof. Let x and y be any pair of distinct points of X and f be injective. Then f(x)≠ f(y) in Y. Since Y is T_0, there exists an open set B containing say f(x) but not f(y). Then, we have x∈f^{-1}(B) and y ∉ f^{-1}(B). Since f is totally $b^\#$-continuous, by Definition 3.6, f^{-1}(B) is $b^\#$-clopen in X. Also x∈f^{-1}(B) and y∈X\f^{-1}(B). This implies every pair of distinct points of X can be separated by disjoint $b^\#$-open sets in X. Therefore by using Definition 3.1(iii) X is $b^\#T_2$. This proves (i).

Now to prove (ii). Let x, y∈X and x ≠ y. Since f is injective, f(x) ≠ f(y) in Y. Further, since Y is T_2, there exist open sets *G* and H in Y such that f(x)∈G, f(y)∈Hand G∩H=ϕ. This implies x∈f^{-1}(G) and y∈f^{-1}(H). Since f is totally $b^\#$-continuous, by Definition 3.6, f^{-1}(G) and f^{-1}(H) are $b^\#$-clopen sets in X. Also f^{-1}(G)∩ f^{-1}(H) = f^{-1}(G∩H) = ϕ. Thus every two distinct points of X can be separated by disjoint $b^\#$-open sets. Therefore by using Definition 3.1(iii), X is $b^\#T_2$.This proves (ii).

Theorem 3.11. If f: (X,τ) → (Y,σ) is $b^\#$-totally continuous injection and Y is $b^\#T_0$, then X is ultra Hausdorff.

Proof. Let x and y be any pair of distinct points of X and f be injective. Then f(x)≠f(y) in Y. Since Y is $b^\#T_0$, by Definition 3.1(i), there exists a $b^\#$-open set B containing say f(x) but not f(y). Then, we have x∈f^{-1}(B) and y ∉ f^{-1}(B). Since f is $b^\#$-totally continuous, by Definition 3.5,f^{-1}(B) is clopen in X. Also x∈f^{-1}(B) and y∈X\f^{-1}(B). This implies every pair of distinct points of X can be separated by disjoint clopen sets in X. Therefore by Definition 2.2(ii), X is ultra Hausdorff.

4. $b^\#$-regular and $b^\#$-normal spaces

In this section we define $b^\#$-regular and $b^\#$-normal spaces and investigate their properties.

Definitions 4.1. Let (X, τ) be a topological space. Then X is said to be

(i) $b^\#$-regular if for every $b^\#$-closed set F and a point x∉F there exists disjoint open sets U and V such that x∈ U and F⊆V.
(ii) $b^\#$-normal if for each pair of disjoint $b^\#$-closed sets A, B of X there exist disjoint open sets U and V such that A⊆ U and B⊆ V.

Example 4.2. Consider the topology τ = {ϕ , {a}, {c}, {c, b, a}, {a, c, d}, {a, b}, {a, c}, X} on X= {a, b, c, d}. Then $b^\#$-O(X, τ) = {ϕ , {c}, {b, a}, X} = $b^\#$-CL (X, τ). This space is $b^\#$-regular and $b^\#$-normal.

Theorem 4.3. In a topological space (X, τ) the following are equivalent.

(i) (X, τ) is $b^\#$-regular.
(ii) For every x∈X and every $b^\#$-open set U containing x, there exist an open set V such that x∈V⊆*cl*(V)⊆U.
(iii) For every $b^\#$-closed set A, the intersection of all the closed neighborhoods of A is A.
(iv) For every non empty set A and a $b^\#$-open set B such that A∩B≠ϕ,there exist an open set F such that A∩F≠ϕ and *cl*(F)⊆ B.
(v) For every non empty set A and $b^\#$-closed set B such that A∩B=ϕ, there exist disjoint open sets L and M such that A∩L≠ϕ and B⊆M.

Proof. Suppose (X,τ) is $b^\#$-regular. Let x∈X and let U be a $b^\#$-open set containing x. Then X\U is $b^\#$-closed. Since (X, τ) is a $b^\#$-regular space, by Definition 4.1(i), there exist open sets V_1 and V_2 such that V_1∩V_2=ϕ and x∈V_1,X\U⊆V_2. Since V_1∩V_2=ϕ, V_1⊆X\$V_2$⊆U. This implies that *cl*(V_1)⊆*cl*(X\$V_2$)=X\$V_2$⊆U. Take V=$V_1$. Therefore x∈V⊆*cl*(V)⊆U. This proves (i)⇒(ii).

Suppose (ii) holds. Let A be $b^\#$-closed and x∉A. Then X\A is $b^\#$-open and x∈X\A. Then by (ii), there exist an open set V such that x∈V⊆*cl*(V)⊆ X\A. Thus A⊆X*cl*(V)⊆ X\V and x∉X\V. Since X\V is a closed neighborhood of A and x∉X\V, it follows that intersection of all closed neighborhoods containing A is A. This proves (ii)⇒ (iii).

Assume (iii). Suppose A∩B≠ϕ and B is $b^\#$-open. Let x∈A∩B. Since B is $b^\#$-open, X\B is $b^\#$-closed and x∉X\B. By our assumption, there exists a closed neighborhood V of X\B such that x∉V. Since V is a neighborhood of X\B, there exists an open set U such that X\B⊆U⊆V. Then F=X\V is closed and x∈F. Since x∈A, A∩F≠ϕ. Also X\U is closed and *cl*(F)=*cl*(X\V)⊆X\U⊆B. This shows that *cl*(F)⊆ B. This proves (iii)⇒(iv).

Assume (iv). Suppose A∩B=ϕ, where A is non-empty and B is $b^\#$-closed. Then X\B is $b^\#$-open and A∩ (X\B) ≠ϕ. By (iv), there exists an open set L such that A∩L≠ϕ and L⊆cl(L)⊆ X\B. Put M=X*cl*(L). Then B ⊆M and L,M are open sets such that M=X*cl*(L)⊆ (X\L). This proves (iv)⇒ (v).

Suppose (v) holds. Let B be $b^\#$-closed and x∉B. Then B∩{x}=ϕ. By (v), there exists disjoint open sets L and M such that L∩{x}≠ϕ and B⊆M. Since L∩{x}≠ϕ, x∈L.

Therefore by using Definition 4.1(i), (X,τ) is $b^{\#}$-regular. This proves (v)⇒(i).

Theorem 4.4. A topological space (X, τ) is $b^{\#}$-regular if and only if every pair consisting of a compact set A and a disjoint $b^{\#}$-closed set B can be separated by $b^{\#}$ - open sets.

Proof. Let (X, τ) be $b^{\#}$-regular and let A be a compact set, B a $b^{\#}$-closed set with $A\cap B=\phi$. Since (X, τ) is $b^{\#}$-regular, by Definition 4.1(i), for each $x\in A$, there exist disjoint open sets U_x and V_x such that $x\in U_x, B\subseteq V_x$. Clearly $\{U_x: x\in A\}$ is an open covering of the compact set A. Since A is compact, there exist a finite sub family $\{U_{xi}: i=1, 2, \ldots n\}$ which covers A. It follows that $A\subseteq\cup\{U_{xi}: i=1, 2, \ldots n\}$ and $B\subseteq\cap\{V_{xi}: i=1, 2, \ldots n\}$. $\{U_{xi}: i=1, 2, \ldots n\}$.

Put $U=\cup\{U_{xi}: i=1, 2, \ldots n\}$ and $V=\cap\{V_{xi}: i=1, 2, \ldots n\}$ then $U\cap V=\phi$. For, if $x\in U\cap V$ then $x\in U_{xj}$ for some j and $x\in V_{xi}$ for every i. This implies that $x\in U_{xi}\cap V_{xi}$, which is a contradiction to $U_{xi}\cap V_{xi}=\phi$. Thus U and V are disjoint open sets containing A and B respectively. Conversely suppose every pair consisting of a compact set A and a disjoint $b^{\#}$-closed set B can be separated by open sets. Let B be any $b^{\#}$-closed set and $x\notin B$. Then $\{x\}$ is a compact sub set of X and $\{x\}\cap B=\phi$. By our assumption, there exist disjoint open sets U and V such that $x\in U$ and $B\subseteq V$. Therefore by using Definition 4.1(i), (X, τ) is $b^{\#}$-regular.

Theorem 4.5. If for each $b^{\#}$-closed set F of (X, τ) and each $x\in X\setminus F$, there exist open sets U and V of (X, τ) such that $x\in U$ and $F\subseteq V$ and $cl(U)\cap cl(V)=\phi$ then (X,τ) is $b^{\#}$-regular.

Proof. Suppose for each $b^{\#}$-closed set F of (X, τ) and each $x\in X\setminus F$, there exist open sets U and V of (X, τ) such that $x\in U$ and $F\subseteq V$ and $cl(U)\cap cl(V)=\phi$. Now $U\cap V\subseteq cl(U)\cap cl(V)=\phi$. This implies that $U\cap V=\phi$. Thus by Definition 4.1(i), (X, τ) is $b^{\#}$-regular.

Theorem 4.6. If a finite topological space (X, τ) is T_2 then it is $b^{\#}$-regular.

Proof. Suppose a finite topological space (X, τ) is T_2. Let S be a $b^{\#}$-closed set and $x\notin S$. Then for each $y\in S$, there exist disjoint open sets U_y and V_y such that $x\in U_y$ and $y\in V_y$. Now $x\in\bigcap_{y\in S}U_y$ and $S\subseteq\bigcup_{y\in S}V_y$. Take $U=\bigcap_{y\in S}U_y$ and $V=\bigcup_{y\in S}V_y$. Since $S\subseteq X$ is finite, U and V are open in X. Also $U\cap V=\phi$. For $z\in U\cap V$ then $z\in U_y$ for each $y\in S$ and $z\in V_y$ for some $y\in S$. This implies $z\in U_y\cap V_y$ for some $y\in S$. This contradicts the fact that $U_y\cap V_y=\phi$. Thus U and V are disjoint open sets containing x and S respectively. Therefore by Definition 4.1(i), (X, τ) is $b^{\#}$-regular.

Theorem 4.7. Let f: $X\rightarrow Y$ be a function.

(i) If f is $b^{\#}$-continuous, open, surjective and X is $b^{\#}$-regular then Y is regular.

(ii) If f is open, $b^{\#}$-irresolute, bijective and X is $b^{\#}$-regular then so is Y.

Proof. Suppose f is $b^{\#}$-continuous, open, surjective function and X is $b^{\#}$-regular. Let $y\in Y$ and let S be closed in Y such that $y\notin S$. Since f is $b^{\#}$-continuous, by Lemma 2.6(i), $f^{-1}(S)$ is $b^{\#}$-closed in X. Since f is surjective, there exists $x\in X$ such that $y=f(x)$. Since $f(x)=y\notin S$ it follows that $x\notin f^{-1}(S)$. Since (X,τ) is $b^{\#}$-regular, by Definition 4.1(i), there exist disjoint open sets U and V such that $x\in U$ and $f^{-1}(S)\subseteq V$. That is $f(x)\in f(U)$ and $f(f^{-1}(S))\subseteq f(V)$. That is $y\in f(U)$ and $S\subseteq f(V)$. Since f is an open map, f(U) and f(V) are open in Y. This establishes that Y is regular. This proves (i).

Suppose X is $b^{\#}$-regular. Let S be $b^{\#}$-closed in Y and $y\in Y\setminus S$. Since f is $b^{\#}$-irresolute, by Lemma 2.6(i), $f^{-1}(S)$ is $b^{\#}$-closed in X. Since f is onto, there exist $x\in X$ such that $y=f(x)$. Now $y\in Y\setminus S$ implies $y=f(x)\notin S\Rightarrow x\notin f^{-1}(S)$. Since X is $b^{\#}$-regular and since $f^{-1}(S)$ is $b^{\#}$-closed in X such that $x\notin f^{-1}(S)$, by Definition 4.1(i),there exist open sets U and V in X such that $U\cap V=\phi$ and $x\in U$, $f^{-1}(S)\subseteq V$. Now $x\in U\Rightarrow f(x)\in f(U)$ and $f^{-1}(S)\subseteq V\Rightarrow S\subseteq f(V)$. Since f is an open map, f(U) and f(V) are disjoint open sets in Y containing y and $f^{-1}(S)$ respectively. Thus by using the Definition 4.1(i), X is $b^{\#}$-regular. This proves (ii).

Theorem 4.8. A topological space(X, τ) is $b^{\#}$-normal if and only if given a $b^{\#}$-closed set A and a $b^{\#}$-open set U containing A, there is an open set V such that $A\subseteq V\subseteq cl(V)\subseteq U$.

Proof. Suppose (X, τ) is $b^{\#}$-normal. Let A be a $b^{\#}$-closed set and U be a $b^{\#}$-open set containing A. Let $B=X\setminus U$. Then B is $b^{\#}$-closed in X and disjoint from A. Since (X, τ) is $b^{\#}$-normal, by Definition 4.1(ii), there exist disjoint open sets V and W containing A and B respectively. Then $cl(V)$ is disjoint from B. Also $cl(V)\subseteq X\setminus B=U$. Conversely, suppose that for given a $b^{\#}$-closed set A and a $b^{\#}$-open set U containing A, there is an open set V such that $A\subseteq V\subseteq cl(V)\subseteq U$. Let B be a $b^{\#}$-closed set in X disjoint from A. Let $U=X\setminus B$. Then U is an $b^{\#}$-open set containing A. By our assumption, there exist an open set V containing A such that $cl(V)\subseteq U$ which implies $B=X\setminus U\subseteq X\setminus cl(V)$. Thus V and $X\setminus cl(V)$ are disjoint open sets containing A and B respectively. Then by Definition 4.1(ii), (X,τ) is $b^{\#}$-normal.

Theorem 4.9. For a space (X, τ) then the following are equivalent.

(i) (X, τ) is $b^{\#}$-normal.

(ii) For every pair of $b^{\#}$-open sets U and V whose union is X, there exist closed sets A and B such that $A\subseteq U$, $B\subseteq V$ and $A\cup B=X$.

(iii) For every $b^{\#}$-closed set F and every $b^{\#}$-open set G containing F, there exist an open set U such that $F\subseteq U\subseteq cl(U)\subseteq G$.

Proof. Suppose (i) holds. Let U and V be a pair of $b^{\#}$-open sets in a $b^{\#}$-normal space X such that $X=U\cup V$. Then $(X\setminus U)\cap(X\setminus V)=X\setminus(U\cup V)=X\setminus X=\phi$ and $X\setminus U$, $X\setminus V$ are disjoint $b^{\#}$-closed sets. Since (X, τ) is $b^{\#}$-normal, by Definition 4.1(ii), there exist disjoint open sets G and H such that $X\setminus U\subseteq G$ and $X\setminus V\subseteq H$. Let $A=X\setminus G$ and $B=X\setminus H$. Then A and B are closed sets such that $A\subseteq U$, $B\subseteq V$. Also $A\cup B=(X\setminus G)\cup(X\setminus H)=X\setminus(G\cap H)=X\setminus\phi=X$. This proves (i)⇒ (ii).

Suppose (ii) holds. Let F be a $b^{\#}$-closed set and let G be a $b^{\#}$-open set containing F. Then $X\setminus F$ and G are $b^{\#}$-open sets whose union is X. Then by (ii), there exist closed sets W_1 and W_2 such that $W_1\subseteq X\setminus F$ and $W_2\subseteq G$ and $W_1\cup W_2=X$. Then $F\subseteq X\setminus W_1$, $X\setminus G\subseteq X\setminus W_2$ and

$(X\setminus W_1)\cap(X\setminus W_2)=X\setminus(W_1\cup W_2)=X\setminus X=\phi$. Let $U=X\setminus W_1$ and $V=X\setminus W_2$. Then U and V are disjoint open sets such that $F\subseteq U\subseteq X\setminus V=W_2\subseteq G$. As $X\setminus V$ is closed, we have $cl(U)\subseteq X\setminus V$ and $F\subseteq U\subseteq cl(U)\subseteq G$. This proves (ii)⇒ (iii).
Suppose (iii) holds. Let F_1 and F_2 be any two disjoint $b^\#$-closed sets in X. Put $G=X\setminus F_2$. Then $F_1\subseteq G$ and G is a $b^\#$-open set. Then by (iii), there exist an open set U of X such that $F_1\subseteq U\subseteq cl(U)\subseteq G$. It follows that $F_2\subseteq X\setminus cl(U)=V$(say). Then V is open and $U\subseteq V\subseteq cl(U)\cap(X\setminus cl(U))=\phi$. Thus F_1 and F_2 are separated by open sets U and V. By Definition 4.1(ii), (X, τ) is $b^\#$-normal. This proves (iii)⇒ (i).

Theorem 4.10. Let f: X→Y be a function.
(i) If f is surjective, open, $b^\#$-continuous and X is $b^\#$-normal then Y is normal.
(ii) If f is open, $b^\#$-irresolute, bijective and X is $b^\#$-normal then Y is $b^\#$-normal.

Proof. Suppose X is $b^\#$-normal. Let A and B be a disjoint closed sets in Y. Since f is $b^\#$-continuous, by using Lemma 2.6(i), $f^{-1}(A)$ and $f^{-1}(B)$ are $b^\#$-closed in X. Since X is $b^\#$-normal, by Definition 4.1(ii), there exist disjoint open sets U and V such that $f^{-1}(A)\subseteq U$ and $f^{-1}(B)\subseteq V$. That is $f(f^{-1}(A))\subseteq f(U)$ and $f(f^{-1}(B))\subseteq f(V)$. That is $A\subseteq f(U)$ and $B\subseteq f(V)$. Since f is an open map, f(U) and f(V) are open in Y such that $A\subseteq f(U)$ and $B\subseteq f(V)$. This proves that Y is normal. This proves (i).
Suppose X is $b^\#$-irresolute. Let A and B be disjoint $b^\#$-closed sets in Y. Since f is $b^\#$-irresolute, by Lemma 2.6(ii), $f^{-1}(A)$ and $f^{-1}(B)$ are $b^\#$-closed in X. Since X is $b^\#$-normal, by Definition 4.1(ii), there are open sets U and V in X such that $U\cap V=\phi$ and $f^{-1}(A)\subseteq U$ and $f^{-1}(B)\subseteq V$. Since f is onto, $f(f^{-1}(A))\subseteq f(U)$ and $f(f^{-1}(B))\subseteq f(V)$. That is $A\subseteq f(U)$ and $B\subseteq f(V)$. Suppose $y\in f(U)\cap f(V)$ which implies y=f(a) for some $a\in U$ and y=f(b) for some $b\in V$. Since f is injective, a=b that contradicts $U\cap V=\phi$. That is f(U) and f(V) are disjoint open sets containing A and B respectively. Therefore by Definition 4.1(ii) implies that Y is $b^\#$-normal. Thus (ii) is proved.

Theorem 4.11. Let f : (X, τ) → (Y, σ) be a totally $b^\#$-continuous injective open function.
(i) If X is $b^\#$- regular then Y is regular.
(ii) If X $b^\#$-normal then Y is normal.

Proof. Let B be a closed set in Y and $y\notin B$. Take y = f(x). Since f is totally $b^\#$-continuous, by Theorem 4.3.6, $f^{-1}(B)$ is $b^\#$-clopen in X. Let $G = f^{-1}(B)$. Then we have $x\notin G$. Since X is $b^\#$-regular, by Definition 4.1(i), there exist disjoint open sets U and V such that $G\subseteq U$ and $x\in V$. This implies $B=f(G)\subseteq f(U)$ and $y = f(x)\in f(V)$. Further, since f is injective and open, we have $f(U)\cap f(V)= f(U\cap V)= f(\phi)= \phi$ and f(U) and f(V) are open sets in Y. Thus, for each closed set B in Y and each $y\notin B$, there exist disjoint open sets f(U) and f(V) in Y such that $B\subseteq f(U)$ and $y\in f(V)$. Therefore Y is regular. This proves (i).
Now to prove (ii). Let G and H be any two disjoint closed sets in Y. Since f is totally $b^\#$-continuous, by Theorem 3.8, $f^{-1}(G)$ and $f^{-1}(H)$ are $b^\#$-clopen subsets of X. Take $U= f^{-1}(G)$ and $V= f^{-1}(H)$. Since f is injective, we have $U\cap V =f^{-1}(G)\cap f^{-1}(H) =f^{-1}(G\cap H) = f^{-1}(\phi) = \phi$. Since X is $b^\#$-normal, by Definition 4.1(ii), there exist disjoint open sets A and B such that $U\subseteq A$ and $V\subseteq B$. This implies $G= f(U)\subseteq f(A)$ and $H= f(V)\subseteq f(B)$. Further, since f is injective open, f(A) and f(B) are disjoint open sets. Thus, each pair of disjoint closed sets in Y can be separated by disjoint open sets. Therefore Y is normal. This proves (ii).

Theorem 4.12. If f : (X,τ) → (Y,σ) is $b^\#$-totally continuous injective open function from a clopen regular space X onto a space Y, then Y is $b^\#$-regular.

Proof. Let B be a $b^\#$-closed set in Y and $y\notin B$. Take y=f(x). Since f is $b^\#$-totally continuous, by Theorem 3.6, $f^{-1}(B)$ is clopen in X. Let $G=f^{-1}(B)$. Then we have $x\notin G$. Since X is clopen regular, by Definition 2.2(iv), there exist disjoint open sets U and V such that $G\subseteq U$ and $x\in V$. This implies $B=f(G)\subseteq f(U)$ and $y=f(x)\in f(V)$. Further, since f is injective and open, we have $f(U)\cap f(V) = f(U\cap V)= f(\phi)=\phi$and f(U) and f(V) are open sets in Y. Thus, for each $b^\#$-closed set Bin Y and each $y\notin B$, there exist disjoint open sets f(U) and f(V) in Y such that $B\subseteq f(U)$ and $y\in f(V)$. Therefore by Definition 4.1(i), Y is $b^\#$-regular.

Definition 4.13. A function f : (X, τ) → (Y, σ) is said to be $b^\#$-open($b^\#$-closed) if f(U) is $b^\#$-open($b^\#$-closed) in Y for each open(closed) set U in X.

Theorem 4.14. Let f: (X, τ) → (Y, σ) be a totally continuous and $b^\#$-closed injection.
(i) If Y is $b^\#$-regular, then X is ultra regular.
(ii) If Y is $b^\#$-normal, then X is ultra normal.

Proof. Let F be a closed set in X not containing x. Since f is $b^\#$-closed, by Definition 4.13, f(F) is a $b^\#$-closed set in Y not containing f(x). Since Y is $b^\#$-regular, by Definition 4.1(i) there exists disjoint open sets A and B such that $f(x)\in A$ and $f(F)\subseteq B$, which implies $x\in f^{-1}(A)$ and $F\subseteq f^{-1}(B)$. Since f is totally continuous, by Definition 2.3, $f^{-1}(A)$ and $f^{-1}(B)$ are clopen sets. Moreover, since f is injective, $f^{-1}(A)\cap f^{-1}(B) = f^{-1}(A\cap B) = f^{-1}(\phi) =\phi$. Thus, for each pair of a point and a closed set not containing the point, they can be separated by disjoint clopen sets. Therefore by Definition 2.2(vi), X is ultra regular. This proves (i).
Now to prove (ii). Let A and B be disjoint closed subsets of X. Since f is $b^\#$-closed and injective, by Definition 4.13, f(A) and f(B) are disjoint $b^\#$-closed subsets of Y. Since Y is $b^\#$-normal, by Definition 4.1(ii), f(A) and f(B) are separated by disjoint open sets U and V respectively. Therefore we obtain, $A\subseteq f^{-1}(U)$ and $B\subseteq f^{-1}(V)$. Since f is totally continuous, by Definition 2.3, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen sets in X. Also, $f^{-1}(U)\cap f^{-1}(V) = f^{-1}(U\cap V) =\phi$. Thus each pair of non empty disjoint closed sets in X can be separated by disjoint clopen sets in X. Therefore by Definition 2.2(iii), X is ultra normal.

Theorem 4.15. If f : (X, τ) → (Y,σ) is a $b^\#$-totally continuous, injective and open function from a clopen normal space X onto a space Y then Y is $b^\#$-normal.

Proof. Let G and H be any two disjoint $b^\#$-closed sets in Y. Since f is $b^\#$-totally continuous, by Theorem 3.6, $f^{-1}(G)$ and $f^{-1}(H)$ are clopen subsets of X. Take $U=f^{-1}(G)$ and $V=f^{-1}(H)$. Since f is injective, we have $U\cap V = f^{-1}(G)\cap f^{-1}(H) =f$

$^{-1}(G \cap H) = f^{-1}(\phi) = \phi$. Since X is clopen normal by Definition 2.2(v), there exist disjoint open sets A and B such that $U \subseteq A$ and $V \subseteq B$. This implies $G=f(U) \subseteq f(A)$ and $H=f(V) \subseteq f(B)$. Further, since f is injective open, f(A) and f(B) are disjoint open sets. Thus, each pair of disjoint $b^{\#}$-closed sets in Y can be separated by disjoint open sets. Therefore by Definition 4.1(ii), Y is $b^{\#}$-normal.

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