International Journal of Applied Research

**ISSN Print:** 2394-7500
**ISSN Online:** 2394-5869
**Impact Factor:** 5.2
IJAR 2015; 1(8): 666-669
www.allresearchjournal.com
Received: 24-05-2015
Accepted: 27-06-2015

**S. Vidhyalakshmi**
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu.

**M.A. Gopalan**
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu.

**R. Presenna**
M. Phil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu.

**N. Christy**
M. Phil Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu.

**Correspondence:**
**S. Vidhyalakshmi**
Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu.

# On The Binary Quadratic Diophantine Equation $x^2 - 3xy + y^2 + 21x = 0$

**S. Vidhyalakshmi, M.A. Gopalan, R. Presenna, N. Christy**

**Abstract**

The binary quadratic equation $x^2 - 3xy + y^2 + 21x = 0$ represents a hyperbola. In this paper we obtain a sequence of its integral solutions and present a few interesting relations among them.

**Keywords:** Binary quadratic equation, Integral solutions.
MSC subject classification: 11D09.

## 1. Introduction

The binary quadratic Diophantine equations (both homogeneous and non homogeneous) are rich in variety [1 – 6]. In [7 – 16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. These results have motivated us to search for infinitely many non-zero integral solutions of an another interesting binary quadratic equation given by $x^2 - 3xy + y^2 + 21x = 0$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

## 3. Method of Analysis:

The Diophantine equation representing the binary quadratic equation to be solved for its non-zero distinct integral solution is

$$x^2 - 3xy + y^2 + 21x = 0 \tag{1}$$

Note that (1) is satisfied by the following non-zero integer pairs
(21,0), (21,21), (21,42), (-21,-63).
However, we have other solutions to (1), which are illustrated below:
Treating (1) as a quadratic in x and solving for x, we have

$$x = \frac{1}{2}\left[(3y-21) \pm \sqrt{5y^2 - 126y + 441}\right] \tag{2}$$

To eliminate the square root on the RHS of (2), assume

$$\alpha^2 = 5y^2 - 126y + 441$$

Multiplying both sides of the above equation by 5 and performing a few simplication, we have

$$Y^2 = 5\alpha^2 + 1764 \quad (3)$$

where

$$Y = 5y - 63 \quad (4)$$

The smallest positive integer solutsion to (3) is

$$\alpha_0 = 6,\ Y_0 = 42$$

Now, to find the other solution of (3), consider the Pellian equation

$$Y^2 = 5\alpha^2 + 1 \quad (5)$$

The general solution is $(\tilde{\alpha}s, \tilde{Y}s)$ of (5) is given by

$$\tilde{Y}_s = \frac{f_s}{2} \qquad \tilde{\alpha}_s = \frac{g_s}{2\sqrt{5}}$$

Where

$$f_s = [(9+4\sqrt{5})^{s+1} + (9-4\sqrt{5})^{s+1}]$$

$$g_s = [(9+4\sqrt{5})^{s+1} - (9-4\sqrt{5})^{s+1}],\ s = -1,1,3,5\ldots\ldots$$

The other values of α and Y solution (3) are obtained by applying Brahmagupta lemma between the solutions $(\alpha_0, Y_0)\ \&\ (\tilde{\alpha}_s, \tilde{Y}_s)$ Thus,

$$\alpha_{s+1} = \frac{21 g_s}{\sqrt{5}} \quad (6)$$

$$Y_{s+1} = 21 f_s \quad (7)$$

Satisfying (6) and (7) in (2) and taking the positive sign before the square root of the RHS of (2), we have

$$x_{s+1} = \frac{1}{2}\left[\frac{1}{5}(21 f_s + 63) + \frac{21 g_s}{\sqrt{5}}\right] \quad (8)$$

$$y_{s+1} = \frac{1}{5}(21 f_s + 63) \quad (9)$$

A few numerical examples are given in Table below:

**Table 1:** Examples

| s | $x_{s+1}$ | $y_{s+1}$ |
|---|---|---|
| -1 | 21 | 21 |
| 1 | 1365 | 3549 |
| 3 | 435477 | 1140069 |
| 5 | 140218197 | 367095981 |
| 7 | 45149819925 | 118203763125 |

The recurrence relations satisfied by $x_{s+1}, y_{s+1}$ are respectively

(i) $x_{s+5} - 322 x_{s+3} + x_{s+1} = -2688$

(ii) $y_{s+5} - 322 y_{s+3} + y_{s+1} = -4032$

Proof of (i):
Now, (8) is written as

$$5 y_{s+1} - 63 = 21 f_s \quad (10)$$

Replace s by s+2 in (10),

$$5 y_{s+3} - 63 = 21 f_{s+2}$$

$$5 y_{s+3} - 63 = 3381 f_s + 1512\sqrt{5} g_s \quad (11)$$

Replace s by s+4 in (10),

$$5 y_{s+5} - 63 = 21 f_{s+4}$$

$$05 y_{s+5} - 63 = 1088661 f_s + 486864\sqrt{5} g_s \quad (12)$$

From (10),(11),and (12) we obtain (i).
Proof of (ii):
We have,

$$2x_{s+1} - 3y_{s+1} + 21 = \frac{21}{\sqrt{5}} g_s \quad (13)$$

Replace s by s+2 in (13),

$$2x_{s+3} - 3y_{s+3} + 21 = \frac{21}{\sqrt{5}} g_{s+2}$$

$$2x_{s+3} - 3y_{s+3} + 21 = \frac{3881}{\sqrt{5}} g_s + 1512 f_s \quad (14)$$

Replace s by s+4 in (13),

$$2x_{s+5} - 3y_{s+5} + 21 = \frac{21}{\sqrt{5}} g_{s+4}$$

$$2x_{s+5} - 3y_{s+5} + 21 = \frac{1088661}{\sqrt{5}} g_s + 486864 \; f_s \qquad (15)$$

From (13),(14),and (15) we obtain (ii).

A few interesting properties satisfied by the solutions of (1) are presented below:

1) $x_{s+1}$ and $y_{s+1}$ are always odd.

2) $x_{s+1} \equiv 0(\text{mod}3)$

3) $y_{s+1} \equiv 0(\text{mod}3)$

4) $y_{s+5} = 322y_{s+3} - y_{s+3} - 4032$

5) $2x_{s+1} = \frac{1}{360}(5y_{s+3} + 275y_{s+1} + 2520)$

6) $2x_{s+3} = 8y_{s+3} - 445y_{s+1} + 5523$

7) $2x_{s+5} = 3y_{s+5} + 70y_{s+3} + 104650y_{s+1} - 1319493$

8) $88555y_{s+3} - 275y_{s+5} - 720x_{s+1} = 1106280$

9) $231840x_{s+1} - 5y_{s+5} - 88555y_{s+1} = 831600$

10) $143282y_{s+3} - 445y_{s+5} + 2x_{s+3} = 1799763$

11) $143282y_{s+1} - 8y_{s+5} + 644x_{s+3} = 1810662$

12) $1036y_{s+5} - 644x_{s+5} + 33697370y_{s+1} = 424594506$

13) $33697370y_{s+3} - 104647y_{s+5} - 2x_{s+5} = 423268293$

14) $4425y_{s+1} - 5760x_{s+1} + 10x_{s+3} = 7455$

15) $320400x_{s+1} + 550x_{s+3} - 4425y_{s+3} = 2640225$

16) $10080x_{s+1} - 2x_{s+5} + 3y_{s+5} + 100800y_{s+1} = 1354773$

17) $3013920x_{s+1} - 22x_{s+5} + 33y_{s+5} - 20160y_{s+3} = 25063143$

18) $70x_{s+3} - 8y_{s+5} + 12y_{s+5} + 434175y_{s+1} = 5471277$

19) $267y_{s+5} - 41860x_{s+3} - 178x_{s+5} + 173670y_{s+3} = 1838487$

20) $441\left(5y_{3s+3} + 63\left(\frac{5}{21}y_{s+1} - 3\right) - 63\right)$ is a cubic interger.

21) $\frac{1}{21}(5y_{2s+2} - 21)$ is a perfect square.

22) Each of the following is a nasty number:

- $14(5y_{2s+2} - 21)$
- $14\left(5y_{3s+3} + 63\left(\frac{5}{21}y_{s+1} - 3\right) - 63\right)$

**Remarkable observations**

**1)** By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the other choices of hyperbolas

❖ $Y^2 = 25920X^2 - 45722880$

where $X = 5y_{s+1} - 63$ and

$Y = 5y_{s+3} + 805y_{s+1} + 10080$

❖ $Y^2 = 2880X^2 - 45722880$

where $X = \frac{1}{441}(5y_{s+1} - 63)^3 - 5y_{3s+3} + 63$

$Y = \frac{1}{1512\sqrt{5}}(5y_{s+3} - 805y_{s+1} + 10080)$

**2)** By considering suitable linear transformations between the solutions of (1), one may get integer solutions for the other choices of parabola

$Y^2 = 544320X - 45722880$

Where $X = 5y_{2s+2} - 21$ and

$Y = 5y_{s+3} + 805y_{s+1} + 10080$

**Note**

Taking the negative sign on the R.H.S of (2), another set of x values are given by

$$x_{s+1} = \frac{1}{2}\left(\frac{1}{5}(63f_s + 64) - \frac{21}{\sqrt{5}} g_s\right)$$

**4. Conclusion**

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct solutions for the non-homogeneous binary quadratic equation. To conclude, one may search for other choices of solutions to the considered binary equation and further, quadratic equations with multi-variables.

**Acknowledgement**
The financial support from the UGC, New Delhi (F.MRP-5122/14 (SERO/UGC) dated March 2014) for a part of this work is gratefully acknowledged.

**References**

1. Banumathy TS. A Modern Introduction to Ancient Indian Mathematics, Wiley Eastern Limited, London, 1995.
2. Carmichael RD. The Theory of Numbers and Diophantine Analysis, Dover Publications, New York, 1950.
3. Dickson LE. History of the Theory of Numbers, Chelsia Publicating Co, New York, 1952, II.
4. Mordell LJ. Diophantine Equations, Acadamic Press, London, 1969.
5. Nigel, Smart P. The Algorithm Resolutions of Diaphantine eqations, Cambridge University, Press, London, 1999.
6. Telang SG. Number theory, Tata Mc Graw-Hill Publishing Company, New Delhi, 1996.
7. Gopalan MA, Parvathy G. Integral Points On The Hyperbola $x^2+4xy+y^2-2x-10y+24=0$ ,, Antarctica J Math. 2010; 1(2):149-155.
8. Gopalan MA, Vidhyalakahmi S, Sumathi G, Lakshmi K. "Integral Pionts On The Hyperbola $x^2+6xy+y^2+40x+8y+40=0$", Bessel J Math. 2010; 2(3):159-164
9. Gopalan MA, Gokila K, Vidhyalakahmi S. "On the Diophantine Equation $x^2+4xy+y^2-2x+2y-6=0$", Acta Ciencia Indica 2007; XXXIIIM(2):567-570.
10. Gopalan MA,Vidhyalakahmi S, Devibala S. On The Diophantine Equation $3x^2+xy=14$,Acta CienciaIndica, (2007), Vol.XXXIII M.No2,P.645-646.
11. Gopalan MA, Janaki G. Observations on $x^2-y^2+x+y+xy=2$, Impact J Sci Tech. 2008; 2(3):14, 3-148.
12. Gopalan MA, Shanmuganadham P, Vijayashankar A. On Binary Quadratic Equation $x^2-5xy+y^2+8x-20y+15=0$, Acta Ciencia Indica 2008; XXXIVM(4):1803-1805c.
13. Gopalan MA, Vidhyalakahmi S, Lakshmi K, Sumathi G, Observation on $3x^2+10xy+4y^2-4x+2y-7=0$ Diophantus J Maths. 2012; 1(2):123-125.
14. Mollion RA. All Solutions of the Diophatine Equations $X^2-DY^2=n$ ,, Far East J Math Sci. 1998; III:257-293.
15. Vidhyalakahmi S, Gopalan MA, Lakshmi K. Observation On The Binary Quadratic Equation $3x^2-8xy+3y^2+2x+2y+6=0$, Scholar Journal of Physics, Mathematics and Statistics. 2014; 1(2):41-45.
16. Vidhyalakahmi S, Gopalan MA, Lakshmi K. Integer Solution of the Binary Quadratic Equation $x^2-5xy+y^2+33x=0$, International Journal of Innovative Science Engineering & Technology. 2014; 1(6):450-453.