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The invariance of the Hankel transform under K-Binomial transform of a sequence

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Abstract
We give a new proof of the invariance of the Hankel transform under the binomial transform of a sequence. Our method of proof leads to three variations of the binomial transform; we call these the k-binomial transforms. We give a simple means of constructing these transforms via a triangle of numbers. We show how the exponential generating function of a sequence changes after our transforms are applied, and we use this to prove that several sequences in the On-Line Encyclopedia of Integer Sequences are related via our transforms.
In the process, we prove three conjectures in the OEIS. Addressing a question of Layman, we then show that the Hankel transform of a sequence is invariant under one of our transforms, and we show how the Hankel transform changes after the other two transforms are applied. Finally, we use these results to determine the Hankel transforms of several integer sequences.

Keywords: Hankel transform, K-Binomial transform.

1. Introduction
Given a sequence A = {a0, a1, . . .}, define the binomial transform B of a sequence A to be the sequence B(A) = {bn}, where bn is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} a_i.$$

Define the Hankel matrix of order n of A to be the (n + 1) × (n + 1) upper left sub-matrix of

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let h_n denote the determinant of the Hankel matrix of order n. Then define the Hankel transform H of A to be the sequence H(A) = {h0, h1, h2, . . . , }. For example, the Hankel matrix of order 3 of the derangement numbers, {Dn} = {1, 0, 1, 2, 9, 44, 265, . . .}, is

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 9 \\ 1 & 2 & 9 & 44 \\ 2 & 9 & 44 & 265 \end{bmatrix}$$

The determinant of this matrix is 144, which is (0!)2 (1!)2 (2!)2 (3!)2.

Flajolet and Radoux have shown that the Hankel transform of the derangement numbers is {Qn i=0(n!)2}. Although the determinants of Hankel matrices had been studied before, the term "Hankel transform" was introduced in 2001 by Layman. Layman proves that the Hankel transform is invariant under the binomial transform, in the sense that H(B(A)) = H(A). He also proves that the Hankel transform is invariant under the invert transform and he asks if there are other transforms for which the Hankel transform is invariant.
This article partly addresses Layman's question. We provide a new proof of the invariance of the Hankel transform under the binomial transform. Our method of proof generalizes to three variations of the binomial transform that we call the k-binomial transform, the rising k-binomial transform, and the falling k-binomial transform. Collectively, we refer to these as the k-binomial transforms.
We give a simple means of constructing these transforms using a triangle of numbers, and we provide combinatorial interpretations of the transforms as well. We show how the exponential generating function of a sequence changes after applying our transforms, and we use these results to prove that several sequences in the On-Line Encyclopedia of Integer Sequences (OEIS) are related by the transforms.
In the process, we prove three conjectures listed in the OEIS concerning the binomial mean transform. Then, giving an answer to Layman's question, we show that the Hankel transform is invariant under the falling k-binomial transform. The Hankel transform is not invariant under the k-binomial transform and the rising k-binomial transform, but we give a formula showing how the Hankel transform changes under these two transforms.
These results, together with our proofs of relationships between sequences in the OEIS, determine the Hankel transforms of several sequences in the OEIS. (Unfortunately, this is some discrepancy in the literature in the definitions of the Hankel determinant and the binomial transform of an integer sequence.
The definitions of the Hankel determinant in Layman and in Ehrenborg are slightly different from each other, and ours is slightly different from both of these. As many sequences of interest begin indexing with 0, we define the Hankel determinant and the Hankel transform so that the first elements in the sequences A and H(A) are indexed by 0.
Layman's definition begins indexing A and H(A) by 1, whereas Ehrenborg's definition results in indexing A beginning with 0 and H(A) beginning with 1. Our definition of the binomial transform is that used by Layman and the OEIS.

The K-Binomial Transforms

We now consider three variations of the binomial transform. All three transforms take two parameters: the input sequence A and a scalar k. The k-binomial transform W of a sequence A is the sequence W(A, k) = {wn}, where wn is given by

$$w_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^n a_i = k^n \sum_{i=0}^{n} \binom{n}{i} a_i, & \text{if } k \neq 0 \text{ or } n \neq 0; \\ a_0, & \text{if } k = 0, n = 0. \end{cases}$$

The rising k-binomial transform R of a sequence A is the sequence R(A, k) = {rn}, where rn is given by

$$r_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^i a_i, & \text{if } k \neq 0; \\ a_0, & \text{if } k = 0. \end{cases}$$

The falling k-binomial transform F of a sequence A is the sequence F(A, k) = {fn}, where fn is given by

$$f_n = \begin{cases} \sum_{i=0}^{n} \binom{n}{i} k^{n-i} a_i, & \text{if } k \neq 0; \\ a_n, & \text{if } k = 0. \end{cases}$$

The case k = 0 must be dealt with separately because 0 0 would occur in the formulas otherwise. Our definitions effectively take 0 0 to be 1. These turn out to be "good" definitions, in the sense that all the results discussed subsequently hold under our definitions for the k = 0 case.
When k = 0, the k-binomial transform of A is the sequence {a0, 0, 0, 0, . . .}, the rising k-binomial transform of A is {a0, a0, a0, . . .}, and the falling k-binomial transform is the identity transform.
The k-binomial transform when k = 1/2 is of special interest; this is the binomial mean transform, sequence A075271. When k is a positive integer, these variations of the binomial transform all have combinatorial interpretations similar to that of the binomial transform, although, unlike the binomial transform, they have a two-dimensional component.
If an represents the number of arrangements of n labeled objects with some property P, then wn represents the number of ways of dividing n objects such that

- In one dimension, the n objects are divided into two groups so that the first group has property P.
- In a second dimension, the n objects are divided into k labeled groups.

The interpretation of the second dimension could be something as simple as a coloring of each object from a choice of k colors, independent of the division of the objects in the first dimension. For example, if the input sequence is the derangement numbers, w_n is the number of ways of dividing n labeled objects into two groups such that the objects in the first group are deranged and each of the n objects has been colored one of k colors, independently of the initial division into two groups.
With this interpretation of wn in mind, rn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and each object in the first group is further placed into one of k labeled groups (e.g., colored using one of k colors). Similarly, fn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P and the objects in the second group are further placed into k labeled groups (e.g., colored using one of k colors).

Invariance of the Hankel Transform under the Binomial Transform

Layman proves that H(B(A)) = H(A) for any sequence A. He does this by showing that the Hankel matrix of order n of B(A) can be obtained by multiplying the Hankel matrix of order n of A by certain upper and lower triangular matrices, each of which have determinant 1.
We present a new proof of this result. Our proof technique suggests generalizations of the binomial transform, which we discuss in subsequent sections. We require the following lemma.
Lemma 1. Given a sequence A = {a0, a1, a2, . . .}, create a triangle of numbers T using the following rule:
1. The left diagonal of the triangle consists of the elements of A.
2. Any number off the left diagonal is the sum of the number

to its left and the number diagonally above it to the left. Then the sequence on the right diagonal is the binomial transform of A. For example, the binomial transform of the derangement numbers is the factorial numbers.

Figure 1 illustrates how the factorial numbers can be generated from the derangement numbers using the triangle described in Lemma 1. Although they do not use the term binomial transform, Lemma 1 is essentially proven by Graham, Knuth, and Patashnik.

We present a different proof, one that allows us to prove similar results for the k-binomial transforms we discuss in subsequent sections.

1
0 1
1 1 2
2 3 4 6
9 11 14 18 24
44 53 64 78 96 120
265 309 362 426 504 600 720

Figure 1: Derangement triangle

Proof.

Let t_n be the n^{th} element on the right diagonal of the triangle. By construction of the triangle, we can see from Figure 2 that the number of times element a_i contributes to the value of t_n is the number of paths from a_i to t_n. To move from a_i to t_n requires n path segments, i of which move directly to the right. Thus there are ways to choose which of the n ordered segments are the rightward-moving segments, and the down segments are completely determined by this choice.

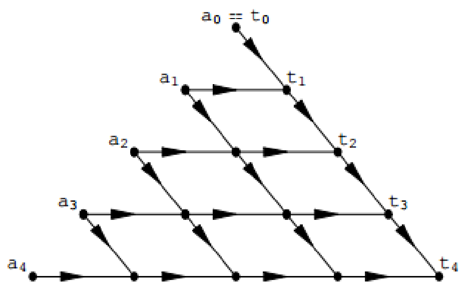

Figure 2: Directed graph underlying the binomial transform

The binomial transform has the following combinatorial interpretation: If an represents the number of arrangements of n labeled objects with some property P, then bn represents the number of ways of dividing n labeled objects into two groups such that the first group has property P.

In terms of the derangement and factorial numbers, then, as Dn is the number of permutations of n ordered objects in which no object remains in its original position, n! is the number of ways that one can divide n labeled objects into two groups, order the objects in the first group, and then permute the first group objects so that none remains in its original position.

The numbers in the triangle described in Lemma 1, not just the right and left diagonals, can also have combinatorial interpretations. For instance, the triangle of numbers in Figure 1 is discussed as a combinatorial entity in its own right in another article by the first author.

The number in row i, position j, in the triangle is the number of permutations of i ordered objects such that every object after j does not remain in its original position. We now give our proof of Layman's result.

Conclusion

We have introduced three generalizations of the binomial transform: the k-binomial transform, the rising k-binomial transform, and the falling k-binomial transform. We have given a simple method for constructing these transforms, and we have given combinatorial interpretations of each of them. We have also shown how the generating function of a sequence changes after applying one of the transforms.

This allows us to prove that several sequences in the On-Line Encyclopedia of Integer Sequences are related by one of these transforms, as well as prove three specific conjectures in the OEIS. In addition, we have shown how the Hankel transform of a sequence changes after applying one of the transforms.

These results determine the Hankel transforms of several sequences listed in the OEIS. We see several areas of further study. One involves continuing to answer Layman's question. We have proved that the Hankel transform is invariant under the falling k-binomial transform, and he proves that the Hankel transform is invariant under the binomial and invert transforms.

Theorem: (Layman) The Hankel transform is invariant under the binomial transform.

Proof.

We define a procedure for transforming the Hankel matrix of order n of a sequence A to the Hankel matrix of order n of B(A) using only matrix row and column addition. While perhaps more complicated than Layman's proof, ours has the virtue of being easily modified to give proofs for the Hankel transforms of the k-binomial transforms that we discuss subsequently.

The procedure is as follows:

1. Given a sequence A = {a0, a1, . . .}, create the triangle of numbers described in Lemma where Ti,j is the (i, j) th entry in the triangle.

2. Let Tn be the following matrix consisting of numbers from the left diagonal of T:

$$\begin{bmatrix} T_{0,0} & T_{1,0} & T_{2,0} & \cdots & T_{n,0} \\ T_{1,0} & T_{2,0} & T_{3,0} & \cdots & T_{n+1,0} \\ T_{2,0} & T_{3,0} & T_{4,0} & \cdots & T_{n+2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n,0} & T_{n+1,0} & T_{n+2,0} & \cdots & T_{2n,0} \end{bmatrix}$$

Since ai = Ti,0, Tn is the Hankel matrix of order n of A.

3. Then apply the following transformations to Tn, where rows and columns of the matrix are indexed beginning with 0. (a) Let i range from 1 to n.

During stage i, for each row j ≥ i, add row j − 1 to row j and replace row j with the result. (b) Then let i again range from 1 to n. During stage i, for each column j ≥ i, add column j − 1 to column j and replace column j with the result.

But this is the Hankel matrix of order n of B(A), as B(A) is the right diagonal of triangle T. Since the only matrix manipulations we used were adding a row to another row and adding a column to another column, and the determinant of a matrix is invariant under these operations, the determinant of the Hankel matrix of order n of A is equal to the determinant of the Hankel matrix of order n of B(A).

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