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On Solving Linear Factorized Quadratic Fractional Programming Problems

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Abstract
A new method namely, *objective fractional separable method* based on simplex method is proposed for finding an optimal solution to a linear factorized quadratic fractional programming (LFQFP) problem in which the numerator of the objective function can be factorized into two linear functions and the denominator of the objective function is of linear type. The solution procedure of the proposed method is illustrated with the numerical example.

Keywords: Linear factorized quadratic fractional programming problem, simplex method, objective fractional separable method.

Introduction
Non-linear programming problem is an optimization problem in which either the objective function is non-linear, or one or more constraints have non-linear relationship or both. The LFQFP is the ratio of linear factorized quadratic function and linear function. Several methods have been developed to solve such problems are proposed. Rashidul Hasan and Babul Hasan [5] have extended the usual simplex method for solving quadratic fractional programming (QFP) problem with inequalities constraints. Archana Khurana and Arora [1] studied for solving a QFP when some of its constraints are homogeneous. Sharma and Jitendra Singh [7] proposed a new approach which is based on the iterative procedure of simplex techniques for solving QFP problems. Nejmaddin and Maher [3, 4] have studied on solving QFP by using the Wolfe's method [8] and a new modified simplex approach. Rashidul Hasan and Babul Hasan [6] have extended simplex method for solving QFP when some of its constraints are homogenous.

2. Preliminaries
The mathematical form of QFP problems is given as follows:

$$\text{Maximize } Z(X) = \frac{\left(C^T X + \delta + \frac{1}{2} X^T G X \right)}{D^T X + \gamma}$$

$$\text{Subject to } AX \left(\leq, =, \geq \right) B, \; X \geq 0$$

Where G are $(n \times n)$ matrix of coefficients with G are symmetric matrices, X is an *n*-dimensional column vector of decision variables, C and D are *n*-dimensional column vector of constants, A is a $(m \times n)$ matrix and B is an *m*-dimensional column vector of constants, δ, γ are scalars.
In this paper, we consider a LFQFP problem in which the numerator of the objective function can be factorized into two linear functions and denominator is of linear type. Such LFQFP problem can be represented as follows:

(P) Maximize $Z(X) = \dfrac{\left(C_1^T X + \alpha\right)\left(C_2^T X + \beta\right)}{\left(D^T X + \gamma\right)}$

Subject to $AX\ (\le, =, \ge)\ B$, $X \ge 0$

Where X is an *n*-dimensional column vector of decision variables, C_1, C_2 and D are *n*-dimensional column vector of constants, A is a $(m \times n)$ matrix and B is an *m*-dimensional column vector of constants, $\alpha, \beta, \delta, \gamma$ are scalars.

Now, from the problem (P), we decompose into two programming problem such that one is of maximization type of numerator objective function (P_1) and the other is of minimization type of denominator objective function (P_2) as constructed below:

(P_1) Maximize $Z_1(X) = \left(C_1^T X + \alpha\right)\left(C_2^T X + \beta\right)$

Subject to $AX\ (\le, =, \ge)\ B$, $X \ge 0$

and

(P_2) Minimize $Z_2(X) = D^T X + \gamma$

Subject to $AX\ (\le, =, \ge)\ B$, $X \ge 0$.

Now, the problem (P_1) can be easily solved by objective separable method proposed by Jayalakshmi and Pandian [2] for finding an optimal solution to a quadratic programming problem in which the objective function can be factorized into two linear functions.

We need the following definitions that can be found in [2].

Definition 2.1 Let $f_1(x)$ and $f_2(x)$ be two differentiable functions defined on $X \subset R^n$, an n-dimensional Euclidean space. The functions $f_1(x)$ and $f_2(x)$ are said to have the Gonzi property in $X \subset R^n$ if

$(f_1(x) - f_1(u))(f_2(x) - f_2(u)) \le 0$, for all $x, u \in X$.

Theorem 2.1 The product $f_1(x) f_2(x)$ of two linear functions $f_1(x)$ and $f_2(x)$ is concave if and only if the functions $f_1(x)$ and $f_2(x)$ has the Gonzi property.

Now, we assume that the two functions $\left(C_1^T X + \alpha\right)$ and $\left(C_2^T X + \beta\right)$ have the Gonzi property in the feasible set and the set of all feasible solutions to the problem (P_1) are non-empty and bounded. Thus, by the Theorem 2.1, it is concluded that the problem (P_1) is a concave non-linear programming problem with linear constraints. This implies that the optimal solution of the problem (P_1) exists and it occurs at an extreme point of the feasible region.

3. Objective Fractional Separable method

Now, we proposed a new method namely, *objective fractional separable method* for finding an optimal solution to the given LFQFP problem.

The proposed method proceeds as follows:

Step 1: Construct (P_1) and (P_2) from the given LFQFP.

Step 2: Solve the problem (P_1) by objective separable method [2]. Let the optimal solution of the problem (P_1) be X_{1° and the Max. $Z_1(X) = Z_1(X_{1^\circ})$.

Step 3: Use the optimal table of the problem (P_1) as an initial simplex table for the problem (P_2), continue to find a sequence of improved basic feasible solutions $\{X_n\}$ to the problem (P_2) and the value of $Z(X)$ at each of the improved basic feasible solution by the simplex method.

Step 4: (a) If $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,..., n-1$ and $Z(X_n) \ge Z(X_{n+1})$ for some n, stop the computation process and then, go to Step 5.

Step 5: (b) If $Z(X_k) \le Z(X_{k+1})$ for all $k = 0,1,2,..., n$ and X_{n+1} is an optimal solution to the problem (D) for some n, stop the computation process and then, go to Step 6.

Step 6: X_n is an optimal solution to the LFQFP problem and Max. $Z(X) = Z(X_n)$.

Step 7: X_{n+1} is an optimal solution to the LFQFP problem and Max. $Z(X) = Z(X_{n+1})$.

Remark 3.1 The maximum value for (*n* +1) is the number of the iterations to obtain the optimal solution to the problem (P_2) by the simplex method.

The proposed method for solving the LFQFP problem is illustrated by the following examples.

Example 3.1 Consider the following LFQFP problem:

(P) Maximize $Z = \dfrac{8x_1^2 + 24x_1x_2 + 18x_2^2 - 2}{6x_1 + 9x_2 + 3}$

Subject to $x_1 + 3x_2 \le 5$;

$2x_1 + x_2 \le 2$; $x_1, x_2 \ge 0$.

Now, the given problem (P) can be written as:

Maximize $Z = \dfrac{(4x_1 + 6x_2 - 2)(2x_1 + 3x_2 + 1)}{(6x_1 + 9x_2 + 3)}$

Subject to $x_1 + 3x_2 \le 5$;

$2x_1 + x_2 \le 2$; $x_1, x_2 \ge 0$.

Using Step 1, construct (P_1) and (P_2) from the above problem (P):

(P_1) Maximize $Z_1(X) = (4x_1 + 6x_2 - 2)(2x_1 + 3x_2 + 1)$

Subject to $x_1 + 3x_2 \le 5; 2x_1 + x_2 \le 2$; $x_1, x_2 \ge 0$.

And

(P_2) Minimize $Z_2(X) = (6x_1 + 9x_2 + 3)$

Subject to $x_1 + 3x_2 \le 5$; $2x_1 + x_2 \le 2$; $x_1, x_2 \ge 0$.

Using Step 2, the optimal solution for the problem (P_1) by Objective separable method is $x_1 = \frac{1}{5}$, $x_2 = \frac{8}{5}$, Maximum $Z_1(X) = \left(\frac{42}{5}\right)\left(\frac{31}{5}\right)$ and the value of

$Z = \frac{1302}{465}$.

Now, by Step 3 of the proposed method, the initial simplex table to the problem (P_2) is given below:

Initial table

	C	6	9	0	0		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution.**	**Ratio**
9	x_2	0	1	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$	4 ←
6	x_1	1	0	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	-
$C_j - Z_{2j}$		0	0	$\frac{12}{5}$ ↑	$\frac{9}{5}$	$Z_2(X) = \frac{93}{5}$	$Z_{\circ 1}(X) = \frac{1302}{465}$

I[st] iteration table: Entering variable is S_1 and leaving variable is X_2.

	C	**6**	**9**	**0**	**0**		
C_B	X_B	x_1	x_2	s_1	s_2	**Solution**	**Ratio**
0	s_1	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	4	
6	x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	
$C_j - Z_{2j}$		0	-6	0	3	$Z_2(X) = 9$	$Z_{\circ 2}(X) = \frac{2}{3}$

Here the I[st] iteration is not optimal, but $Z_{\circ 1} > Z_{\circ 2}$ and by Step 4(a) of the proposed method, the optimal solution to the given LFQFP problem is $x_1 = \frac{1}{5}$, $x_2 = \frac{8}{5}$ and the

Maximum value of $Z(X) = \frac{1302}{465}$.

Example 3.2 Consider the following LFQFP problem:

(P) Maximize $Z = \frac{4x_1^2 + 12x_1x_2 + 8x_2^2 + 4x_1 + 4x_2}{4x_1 + 8x_2 + 4}$

Subject to $-2x_1 + 1x_2 \le 3$;

$4x_1 + 2x_2 \le 8$; $x_1, x_2 \ge 0$.

Now, the given problem (P) can be written as:

Maximize $Z = \frac{(2x_1 + 2x_2)(2x_1 + 4x_2 + 2)}{(4x_1 + 8x_2 + 4)}$

Subject to $-2x_1 + 1x_2 \le 3$;

$4x_1 + 2x_2 \le 8$; $x_1, x_2 \ge 0$.

Using Step 1, construct (P_1) and (P_2) from the above problem (P):

(P_1) Maximize $Z_1(X) = (2x_1 + 2x_2)(2x_1 + 4x_2 + 2)$

subject to $-2x_1 + 1x_2 \leq 3$; $4x_1 + 2x_2 \leq 8$; $x_1, x_2 \geq 0$.

And

(P_2) Minimize $Z_2(X) = (6x_1 + 9x_2 + 3)$

subject to $-2x_1 + 1x_2 \leq 3$; $4x_1 + 2x_2 \leq 8$; $x_1, x_2 \geq 0$.

Now, the optimal solution for the problem (P_1) by objective separable method is $x_1 = \frac{1}{4}$, $x_2 = \frac{7}{2}$, Maximum $Z_1(X) = \left(\frac{15}{2}\right)\left(\frac{33}{2}\right)$ and the value of $Z(X) = \frac{495}{76}$.

Now, by the Step 3 of the proposed method, the solution to the problem (P_2) by simplex method is given below:

Iterations	Optimal Solutions (x_1, x_2, s_1, s_2)	Objective Value of $Z_1(X)$	Objective Value of $Z_2(X)$	Objective Value of $Z(X)$
1	$X_0 = \left(\frac{1}{4}, \frac{7}{2}, 0, 0\right)$	$\left(\frac{15}{2}\right)\left(\frac{33}{2}\right)$	19	$\frac{495}{76}$
2	$X_1 = (2, 0, 7, 0)$	$(4)(6)$	12	2

Here the 2nd iteration is not optimal, but $Z_{\circ 1} > Z_{\circ 2}$ and by Step 4(a) of the proposed method, the optimal solution to the given LFQFP problem is $x_1 = \frac{1}{4}$, $x_2 = \frac{7}{2}$ and Maximum value of $Z(X) = \frac{495}{76}$.

Remark 3.2 The solution of the Example 1 and 2 is same as Nejmaddin and Maher (2013), but they defined a new modified simplex method to solve LFQFP problem, where else the solution is obtained only by using simplex method.

4. Conclusion

The objective fractional separable method is proposed to solve LFQFP problems in which the numerator of the objective function can be factorized into two linear functions and denominator of the objective function is of linear type. Since the proposed method is based on simplex method so it can be easy to compute and to apply. Also, we can solve such LFQFP problems using the existing LP solvers. Further, the present work can be extended to integer and fuzzy LFQFP problems.

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