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# Different phases of Reynolds equation

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**Abstract**
In the theory of hydrodynamic lubrication, two dimensional classical theories were first given by Osborne Reynolds. In 1886, in the wake of a classical Beauchamp Tower's experiment given by Reynolds, he formulated an important differential equation, which was known as: Reynolds Equation given by Reynolds in 1886. Later Osborne Reynolds himself derived an improved version of Reynolds Equation known as Generalized Reynolds Equation, which depends on density, viscosity, film thickness, surface and transverse velocities. The concept of rotation was discussed by Banerjee et al. in 1981 that the rotation of the fluid film which lies across the film gives some new results in lubrication problems of fluid mechanics. The equations for motion of first order rotatory theory and second order rotatory are derived, which have given very important and useful results for journal and thrust bearings.

**Keywords:** Lubrication theory, Reynolds equation, Rotation number

## 1. Introduction
In 1886, Osborne Reynolds presented the basic equation governing the analysis of fluid film lubrication. This equation was the combination of equation of motion and equation of continuity. While deriving this equation Reynolds neglected fluid inertia and gravitational effects in relation to viscous action restricting his analysis to a thin film of iso-viscous and incompressible fluid. However, the application of hydrodynamic theory within the assumptions made by O. Reynolds is valid only over a much narrower field than is generally considered [Halton (1958)]; in particular, high speed variable viscosity, the roughness of surface, thermal effects etc. for emphasizing the need of generalizing the Reynolds equation. Moreover, the increased severity of the bearing operating conditions, the porous bearing and several limitations pertaining to lubricant properties etc. have also necessitated the generalization of Reynolds equation to pertaining the various effects. For the sake of completeness there is a discussion about the gradual development of the Reynolds equation incorporating the various effects which come to be accounted for and gradual relaxing of the assumptions taken in the detailed investigations for approaching the more realistic analysis.

### 1.1 Inertia and Turbulence Effects in Lubrication
In most of the hydrodynamic bearings the lubricant flow is laminar and so that it is governed by the Navier-Stokes equations that relate the pressure and viscous forces acting on the lubricant to the inertia. The inertia terms relative to the viscous terms in this equation can be characterized by a parameter i.e., Reynolds number which increases as the inertia effect increases. In the operation of most bearings the Reynolds numbers are small enough, so that the inertia effect can be neglected, reducing the governing relationship to the familiar Reynolds equation. So that with the continuing trends in machine design for which higher speeds as well as the use of unconventional lubricants such as water of liquid metals, the question of how important inertia will be at high Reynolds numbers in the laminar regime itself is much interesting [11, 19]. If the Reynolds number becomes sufficiently high then the turbulence effect may develop and the previous governing equation will not be applicable even with the inertia terms included. Several contributions towards including the inertia effects and their re-examinations have been made [4, 12, 13, 14]. Inertia effects are also important due to the current interest in assessing the importance of possible visco-elastic effects in the behavior of the lubricant. Both inertial effects and visco-elasticity are important in highly unsteady conditions. Therefore, it is necessary to have a thorough understanding of inertia effects in order to adequately isolate them as not to inadvertently attribute them to visco-

elasticity. The importance of the various fluid inertia terms in the inertia force has been based on the order of magnitude analysis relatively and consequently with varying degree of approximations for simplifying the analysis; various studies have been made [15, 18]. The retention of the inertial terms in the Navier-Stokes equations gives rise to non-linearity, resulting in the analysis becoming much complicated. So, the method of averaged inertia and the method of iteration have been used to account for inertia terms [11, 19]. In 1970, Tichy and Winer used the method of regular perturbation taking Reynolds number as the perturbation parameter, while in 1971, Rodkiewicz and Anwar used a series expansion method.

### 1.2 Thermal Effects in Lubrication

Research into the thermodynamics of fluid film bearing got an impetus by the experimental work of Fogg in 1946, who observed that a parallel surface thrust bearing can support a load and gave an explanation for this by introducing the concept of thermal wedge i.e., the expansion of fluid due to the heating effect. In 1949, Cope modified the classical Reynolds equation by inserting the viscosity and density variation with the fluid film and obtained the respective energy equation for determining the temperature in the film. He coupled the energy balance equation in the fluid film with the momentum and continuity equations to form the temperature and the pressure distribution. Now, the effects of viscosity variation due to pressure and temperature on the characteristics of the slider bearing have been studied by [13]. The variation of viscosity across the fluid film thickness has been analyzed by Cameron [5]. They concluded that the temperature gradients and viscosity variation across the fluid film should not be neglected. In 1962, Dowson generalized the Reynolds equation by taking into consideration variation of fluid characteristics across the fluid film thickness. In 1962, Dowson studied the case of parallel surfaces by taking into account the heat transfer to the bearing surfaces. They found that their investigation completely reversed the earlier predictions of the thermal and viscosity wedge effects. In 1967, Gould discussed the thermo-hydrodynamic performance of fluid between two surfaces approaching each other at a constant velocity. Mc. Callion gave a thermo-hydrodynamic analysis of a finite journal bearing in 1971. The analysis indicated that the bearing load carrying capacity is generally insensitive to heat transfer in the solids. This also showed that the thermo-hydrodynamic load is not bounded by either the adiabatic or the isothermal solutions necessarily.

### 1.3 Non-Newtonian Lubricants

For the development of better, it was found that by adding polymer additives to the base lubricant, the viscosity of the fluid increases and is relatively independent of temperature. This increase in viscosity brings about an increase in load carrying capacity. It was analyzed that the viscosity of the modified lubricant is not constant, but decreases as the rate of strain increases. In 1960, Steidler and Horowitz analyzed the effect of non-Newtonian lubrication on the slider bearing with side leakage. The corresponding experimental analysis was carried out by Dubois et al. in 1960. By using a perturbation technique and choosing n=3, Saibel gave a solution of the slider bearing with side leakage in 1962 and found that the load carrying capacity is reduced by approximately ten percent.

## 2. Mathematical Modelling of a Bearing System

The mathematical modelling of the bearing system is linked with the researches in the field of fluid mechanics of real fluids that started in nineteenth century. The hydrodynamic film lubrication was effectively used before it was scientifically understood. The process of lubrication is a part of the phenomena of hydrodynamics whose scientific analysis was started during nineteenth century. Adams first attempted, developed and patented several rather good designs for railway axle bearing in 1847. The understanding of hydrodynamic lubrication began with the classical Beuchamp Tower's experiments in 1883, with the investigation of friction of the railway partial journal bearing when he observed and calculated the lubricant pressure in the bearing. Later in 1886, Osborne Reynolds derived and employed an equation for the analysis of fluid film lubrication that was a basic governing equation and is named after him as Reynolds equation. He has combined Navier-Stokes equations with equation of continuity to give a second order differential equation for lubricant pressure. This equation is derived under certain assumptions i.e., by neglecting the effects of inertia and gravitational forces in comparison to viscous action, lubricant film to be a thin one of iso-viscous incompressible fluid etc. The conventional Reynolds equation contains viscosity, density and film thickness as parameters. These parameters determine and depend on the temperature and the pressure fields and on the elastic nature of the bearing surfaces. Except these parameters, sometimes surface roughness, porosity and other increased severity of bearing operating conditions etc. may demand the need to generalize the Reynolds equation. Hence the study of hydrodynamic lubrication theory is the form of a study of a particular form of Navier-Stokes equations compatible with the system. From that time, researches in the field of lubrication have made much developments and with the rapid advancement of machines, manufacturing process and materials in which lubrication bears an important role, the study of lubrication has gained much importance and has become, from analytical view point, an independent branch of the fluid mechanics. From practical point of view it remains a section of Tribology. The mathematical modelling of the bearing system consists of various conservation laws of fluid mechanics i.e., mass conservation, momentum, energy and equations of various aspects characterizing the bearing problem such as constitutive equation of lubricant, viscosity dependence on pressure, temperature, equation of state, elastic deformations, surface roughness etc.

### 2.1 Equation of State

For considering the phenomena, it is required to specify the state of fluid which is given by an equation known as equation of state. For an incompressible fluid it is given by

$$\rho = \text{constant} \quad (1)$$

While for a perfect gas for isothermal variations in pressure it is given by
Boyle-Mariotte law as

$$P = \rho R T \quad (2)$$

Here R is the universal gas constant.
For constant compressibility fluids under isothermal conditions, equation of state is written as:

$$\rho = \rho_0 e^{C(P-P_0)} \quad (3)$$

Where $\rho_0$ is the value of ρ at the reference atmospheric pressure $P_0$ and C is the compressibility. This particular equation of state applies rather well to most liquids. For ideal gas flow under adiabatic condition, the equation of state is

$$\rho = \rho_0 \left(\frac{P}{P_0}\right)^{\gamma}$$

Where $\gamma = C_P/C_V$ is the ratio of specific heats at constant pressure and constant volume of the fluid.

For many thermodynamic processes the equation (4) is written as

$$\rho = \rho_0 \left(\frac{P}{P_0}\right)^{n} \quad (5)$$

Where $1 \leq n \leq \gamma$, the equation (5) is called polytropic law.

**2.2 Constitutive Equation of Lubricant**

The equation that relates the viscous contribution to the stress tensor with the rate of deformation tensor is known as constitutive equation applicable to the description of rheological behavior of the lubricant. The constitutive equations are of three kinds i.e., integral type, rate type, and differential type. Mostly lubricating fluids are Newtonian and in such case, shearing stress is directly proportional to rate of strain tensor, constant of proportionality being the dynamic viscosity of the lubricant being Newtonian in character greatly simplifies the analysis. The lubricants which constitute a relationship other than that exists for a Newtonian lubricant are generally known as non-Newtonian lubricant. In case of ideal plastic like grease, some initial stress must be implemented upon the fluid lubricant before the flow begins. The minimum stress necessary to cause the flow is called yield stress. A real plastic behaves in a non-Newtonian type up to certain shearing stress and then starts to behave as if it is Newtonian. A number of fluids have been classified on the basis of their constitutive equation, and had given various names i.e., Maxwell fluid, second order fluid, Walter's fluid, Oldroyd fluid, etc. The class of non-Newtonian fluids for which stress tensor is expressed as directly proportional to some power *n* of deformation tensor is known as power law fluid. If n >1, the fluid is called dilatant and for n <1, it is said to be psudo-plastic. There are some lubricants called thixotropic and in the case of viscosity decreasing with time they are called rheopectic. Lai, Kuei and Mow (1978) have given a list of constitutive equations for the rheological behavior of the synovial fluid which acts as lubricant in synovial joints of human body. Such lubricating fluid properties are viscosity, density, specific heat and thermal conductivity. Among these fluid properties, viscosity plays a more prominent role. Viscosity varies with temperature as well as pressure and this variation plays an important role in lubrication mechanics. The viscosity of heavily loaded lubricating film is treated as a function of both pressure and temperature. As analyzed by Barus (1893) viscosity may be approximated for limited ranges by

$$\mu = \mu_0 e^{[a(P-P_0)+b(T-T_0)]} \quad (6)$$

Where *a* and *b* are said as pressure and temperature viscosity coefficient and the subscript '0' refers to atmospheric conditions. Over reasonably large, ranges of temperature and pressure, the linear relation is taken as:

$$\mu = \mu_0 e^{[1+a(P-P_0)+b(T-T_0)]} \quad (7)$$

**2.3 Continuity Equation**

The fluid flow problems satisfy the basic law of conservation of mass. The equation expressing the law of conservation of mass is called as the continuity equation. This expresses the condition that for any fixed volume of source sink free region, the mass of entering fluid must equal the mass of fluid leaving plus accumulated mass. If the fluid is compressible then the continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\bar{q}) = 0 \quad (8)$$

Where $\bar{q}$ is the velocity vector of the flowing fluid and ρ is the density. If the flow is steady then

$$\frac{\partial\rho}{\partial t} = 0 \quad (9)$$

Hence the continuity equation takes the form

$$\nabla.(\rho\bar{q}) = 0 \quad (10)$$

The equation of continuity for homogeneous, incompressible fluid can be of the form

$$\nabla.(\bar{q}) = 0 \quad (11)$$

On comparing the above equations, we get that the density of the fluid does not appear in the continuity equation for incompressible fluids whereas it does appear in the corresponding equation for compressible fluids. So that the continuity equation for incompressible fluids is a kinematical equation and it is a dynamical for compressible fluid.

**2.4 The Equations of Motion**

The principle of conservation of momentum when applied to fluid contained in a given volume states that the forces acting on the fluid in the control volume equal the rate of outflow of momentum from the control volume through the closed surface enclosing it. The equation expressing this condition for Newtonian, iso-viscous, laminar, continuum and compressible fluid flow for which body forces like gravitational forces or electro- magnetic forces, are considered negligible can takes the form

$$\rho\frac{\partial\bar{q}}{\partial t} + \rho(\bar{q}.\nabla)\bar{q} = -\nabla p + (\lambda+\mu)\nabla(\nabla.\bar{q}) + \mu\nabla^2\bar{q} \quad (12)$$

Where μ is called the coefficient of shear viscosity of the fluid and λ is called coefficient of bulk viscosity. It is often assumed that they are related by 3λ+2μ=0. The above equations (12) were first obtained by Navier in 1821 and later independently by Stokes in 1845. Hence these are known as Navier-Stokes equations. The first term on the left hand side of equation is temporal acceleration term while the second is convective inertia term. The first term on right hand side is due to pressure and the other terms are viscous forces. If, however, the fluid is incompressible, as is the case with most liquid lubricants, then

$$\rho\frac{\partial\bar{q}}{\partial t} + \rho(\bar{q}.\nabla)\bar{q} = -\nabla p + \mu\nabla^2\bar{q}$$

When a large external electro-magnetic field through the electrically conducting lubricant is applied then it gives rise to induced circulating currents, which in turn interacts with the magnetic field and creates a body force called Lorentz force. This extra electro-magnetic pressure pumps the fluid between the bearing surfaces. In that case the Navier-Stokes equations for an incompressible iso-viscous liquid get the form

$$\rho \frac{\partial \bar{q}}{\partial t} + \rho(\bar{q}.\nabla)\bar{q} = -\nabla p + \mu \nabla^2 \bar{q} + \bar{J} \text{x} \bar{B} \quad (13)$$

Where $\bar{J}$ is the electric current density and $\bar{B}$ is the magnetic induction vector. In this case the Maxwell's equations and Ohm's law should also to be considered. These can be

$$\nabla \text{ x } \bar{B} = \mu_0 J \text{ , } \nabla.\bar{B} = 0 \quad (14)$$

$$\bar{J} = \sigma[\bar{E} + \bar{q} \text{x } \bar{B}] \quad (15)$$

$$\nabla \text{ x } \bar{E} = 0 \text{ , } \nabla.\bar{E} = 0 \quad (16)$$

Where $\bar{E}$ is the electric field intensity vector, $\sigma$ is the electrical conductivity and $\mu_0$ is the magnetic permeability of the lubricant.

### 2.5 Energy Equation
This equation gives the conservation of energy principle for a fluid element in which there is no heat source or sink. The equation is given by

$$\rho g \left\{ \frac{\partial (C_V T)}{\partial t} + (\bar{q}.\nabla)(C_V T) \right\} + P\nabla.\bar{q} = \nabla.(K\ \nabla.T) + \phi \quad (17)$$

Where
$C_V$ = Specific heat at constant volume per unit weight,
$T$ = Absolute temperature,
$K$ = Coefficient of thermal conductivity,
$\phi$= Viscous dissipation function.

The above equation of energy states that within an element volume the rate of change of internal energy with compression work must be balanced by the energy conducted by the fluid and dissipated by friction. The first term on the left hand side denotes adiabatic compression work and the first term on right hand side is the conductive heat transfer. For incompressible fluid flow of lubricant adiabatic compression work energy will be zero. The energy equation can't be applied when the film lubricated bearing during the process is isothermal.

### 2.6 Elastic Considerations
For some cases, the thermal stresses or high loading of the bearing surfaces may distort the film shape and it may affect the pressure distribution. The study of this phenomenon of lubrication is called elasto-hydrodynamic lubrication. The effect is important in the lubrication of gear and roller bearings where very high pressure can be developed. The model of a system that involves the interaction of elastic and fluid phenomena, an additional equation to account for elastic deformations is required. This equation is called elasticity equation and it relates the displacements of the solid surface to the stress system.

### 3. Boundary Conditions
The mathematical description of lubrication problem formulated in the above cases is in the form of differential equations which is to be solved and hence until a set of boundary conditions compatible with the physical system is not given, the mathematical description of the model can't be completed. If the film conditions are steady then the momentum equations will be combined with the continuity equation to form an equation that governs the film pressure then it is called Reynolds equation. This is a single differential equation which relates pressure, density, surface velocities and film thickness. It is important to find the boundary conditions before integrating this equation. So the combination of momentum and continuity equations allows the distributed film velocity to be eliminated and replaced by film surface velocities. If unless slip is present these film surface velocities are identical to the velocities of the adjacent bearing surfaces and this condition is known as no-slip condition. For the values of Knudsen number, $K_n$ <0.01, flow may be taken as continuum and no-slip conditions may applied. When 0.01< $K_n$ <15, slip flow becomes significant and for $K_n$> 15, the fully developed molecular flow results will be obtained. With regarding the boundary conditions for the film pressure governed by Reynolds equation the entrance and exit effects of a self acting slider bearing may usually be neglected. So the pressure is taken to be ambient along the boundary. But the hydrodynamic pressures generated in the film are very vast as compared to the ambient pressures, and then the pressure conditions are taken to be zero. So the pressure condition at the source inlet of an externally pressurized film is given by the value of the pressure supplied. Hence the boundary conditions for a particular problem depend upon the peculiarities of the particular situation.

### 4. Basic Assumptions of Hydrodynamic Lubrication
The general mathematical model as given is nonlinear in character besides being a coupled one. So that the severe complexity of the system that describing the problem of lubrication does not lend it at all straight to analysis. Hence, number of simplifications resulting from the physical considerations compatible with the system is required to be made before proceeding to solve the system. The solutions may be of great importance if their limitations are clearly defined. So, it is important that all the assumptions and simplifications be justified and that the limitations imposed shall be understood in analyzing the results. In certain cases, a lot of ideas may be required to be made and then the limits of their applications must be recognized. The order of magnitude analysis may be attempted to analyze the relative effects of various terms in the equations and then to simplify it. The assumptions that are to be made and the simplifications resulting there from could depend upon the nature and the aspect of the problem to be studied. For the analysis which follows to derive the modified Reynolds equation, the following assumptions are taken into consideration, as follows:

1. The lubricant is considered to be incompressible, non-conducting and non-magnetic with constant density and viscosity, unless otherwise given. Most lubricating fluids will satisfy this condition.
2. Flow of the lubricant is laminar, unless otherwise given. A moderate velocity combined with a high kinematic viscosity gives rise to a low Reynolds number, where flow essentially remains laminar.
3. The body forces are neglected, i.e. there are no external fields of force acting on the fluid.

4. Flow is taken as steady, unless otherwise stated, i.e. velocities and fluid properties do not vary with time.
5. Boundary layer is supposed to be fully developed throughout the lubricating region so that entrance effects at the leading edge and the film discontinuity at the trailing edge from which vortices may be shed, are not considered.
6. The fundamental assumption of hydrodynamic lubrication is that the thickness of the fluid is considered very small in comparison with the dimensions of the bearings. Hence the curvature of the film, fluid inertia with respect to viscous forces and variation of pressure along the transverse direction may also be neglected.
7. The fluid behaves as a continuum that implies the pressure is high enough so that the mean free path of the molecule of the fluid are much lower than the effective pore diameter or any other dimension. Hence no slip boundary condition is applicable at the bearing surfaces.
8. The lubricant film is supposed to be iso-viscous.
9. The changes in temperature of the lubricant are not considered.

**5. Modified Reynolds Equation**

The differential equation which is obtained by making use of the assumptions of hydrodynamic lubrication in equations of motion and continuity equation and combining them into a single equation governing lubricant pressure is said to be Reynolds equation. The Reynolds equation when derived for general cases like porous bearings or hydro-magnetic bearings or bearings working with non-Newtonian or magnetic lubricant etc., is called generalized Reynolds equation. This equation is the basic differential equation for the problems of hydrodynamic lubrication. The differential equation was originally derived by Reynolds is restricted to incompressible fluids. The two dimensional classical theory [4, 10] of hydrodynamic lubrication was first given by *Osborne Reynolds* [11]. In 1886, in the wake of a classical experiment by *Beauchamp Tower* [12], he formulated a differential equation known as: Reynolds Equation [11]. The formation and basic mechanism of fluid film was analyzed by that experiment by taking some assumptions:

(i) The film thickness is very small in comparison to the axial and longitudinal dimensions of fluid film.

(ii) If the lubricant layer is to transmit pressure between the shaft and the bearing, the layer must have varying thickness.

*Osborne Reynolds* himself derived "Generalized Reynolds Equation" [7, 10], which depends on density viscosity, film thickness, surface and transverse velocities. The differential equation originally derived by Reynolds was restricted to incompressible fluids, so it was formulated broadly enough to include the effects of compressibility and dynamic loading and was said to be Generalized Reynolds Equation. Thus the final form of Generalized Reynolds Equation [10, 12] was as given:

$$\left(\frac{\rho h^3}{\mu}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{\mu}\frac{\partial P}{\partial z}\right)=6(U_1-U_2)\frac{\partial(\rho h)}{\partial x}+12\rho V \quad (18)$$

Where $\rho$ is the density, $\mu$ is the viscosity, $h$ is the film thickness of fluid film, $U_1$ and $U_2$ are the surface velocities and $V$ is the general velocity. In the equation (1), the term $\{6(U_1-U_2)\frac{\partial(\rho h)}{\partial x}\}$ was due to the bearing velocities along the lubricant film and depends on whether the bearing surfaces have angular or translational velocities, whereas the term $\{12\rho V\}$ was due to relative velocity of bearing surfaces in the direction normal to the fluid film. In most cases, the bearing is stationary and only the runner in thrust bearings and the shaft in the journal bearings are moving, so $U_1=U$ and $U_2=0$. Now the final equation for incompressible lubricants was founded by Reynolds is as given:

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\mu}\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\rho h^3}{\mu}\frac{\partial P}{\partial z}\right)=6U\frac{\partial(\rho h)}{\partial x}+12\rho V_0 \quad (19)$$

Where $U$ is the sliding velocity, $V_0$ is the motion of journal center.

The rotation [1] of fluid film about an axis that lies across the film gives some new results in lubrication problems. The origin of rotation can be traced by certain general theorems related to vorticity in the rotating fluid dynamics. The rotation induces a component of vorticity in the direction of rotation and the effects arising from it are predominant, for large Taylor's number, it results in the streamlines becoming confined to plane transverse to the direction of rotation. The extended version of "Generalized Reynolds Equation" [7, 10] is said to be "Extended Generalized Reynolds Equation" [1, 3] given by *Banerjee et al.*, which takes into account of the effects of the uniform rotation about an axis that lies across the fluid film and depends on the rotation number $M$ [1], i.e. the square root of the conventional Taylor's number. This generalization of the classical theory is known as the "Rotatory Theory of Hydrodynamic Lubrication" [1, 3]. The *Banerjee et al.* [1, 3] had given the derivation of the extended generalized Reynolds equation that is given as follows:

They have considered a layer of fluid rotating at constant rate with rotating angular velocity ω about $z$-axis. Then the equations for momentum and continuity become

$$\rho\frac{\partial u_i}{\partial t}+\rho\frac{\partial u_i}{\partial x_j}=\rho F_i-\frac{\partial}{\partial x_i}\left\{p-\left(\frac{1}{2}\rho\varepsilon_{ijk}\omega_i r_k\right)^2\right\}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\right\}+2\rho\varepsilon_{ijk}u_j x_k \quad (20)$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_j}=0 \quad (20)$$

By using the standard assumption for the lubrication theory [6], they get the reduced equation of momentum be

$$-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}+2\rho\omega v=0 \quad (21)$$

$$-\frac{\partial P}{\partial y}+\mu\frac{\partial^2 v}{\partial z^2}-2\rho\omega u=0 \quad (22)$$

$$-\frac{\partial P}{\partial z} = 0 \tag{23}$$

For the steady flow, the equation of continuity will be

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{24}$$

Where *P* is the modified pressure given by

$$P = p - \left(\frac{1}{2}\rho\varepsilon_{ijk}\omega_i r_k\right)^2 \tag{25}$$

From these equations of momentum, we have

$$\frac{\partial^4 u}{\partial z^4} + \left(\frac{2\rho\omega}{\mu}\right)^2 u = -\frac{2\rho\omega}{\mu^2}\frac{\partial P}{\partial y} \tag{26}$$

$$\frac{\partial^4 v}{\partial z^4} + \left(\frac{2\rho\omega}{\mu}\right)^2 v = -\frac{2\rho\omega}{\mu^2}\frac{\partial P}{\partial x} \tag{27}$$

The boundary conditions on *u* and *v* are

$u=U_0$ at $z=0$ and $u=U_h$ at $z=h$ (28)

$v=0$ at $z=0$ and $z=h$ (29)

By using the boundary conditions, we have

$$\mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} \tag{30}$$

$$\mu\frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial y} + 2\rho\omega U_0 \; at \; z = 0 \tag{31}$$

$$\mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + 2\rho\omega U_h \; at \; z = h \tag{32}$$

Using the non-dimensionless quantities used by *Banerjee et.al. [3]* as given

$$\frac{x}{h_c} = \bar{x}, \frac{u}{q_c} = \bar{u}, \frac{\mu}{\mu_c} = \bar{\mu}, \frac{y}{h_c} = \bar{y}, \frac{v}{q_c} = \bar{v}, \frac{z}{h_c} = \bar{z}, \frac{w}{q_c} = \bar{w}, \frac{h}{h_c} = \bar{h}, \frac{\rho}{\rho_c} = \bar{\rho}, \frac{2\omega {h_c}^2\rho_c}{\mu_c} = M \tag{33}$$

Now dropping the bars for convenience, the equations have reduced to

$$\frac{\partial^4 u}{\partial z^4} + \frac{M^2\rho^2}{\mu^2}u = -\frac{M\rho}{\mu^2}\frac{\partial P}{\partial y} \tag{34}$$

$$\frac{\partial^4 v}{\partial z^4} + \frac{M^2\rho^2}{\mu^2}v = \frac{M\rho}{\mu^2}\frac{\partial P}{\partial x} \tag{35}$$

$$\frac{U_0}{q_c} = U_1 \; at \; z = 0, \frac{U_h}{q_c} = U_2 \; at \; z = h \tag{36}$$

Also $v=0$ at $z=0$ and $z=h$, the expressions are

$$\mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + M\rho U_1 \tag{37}$$

$$\mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial P}{\partial y} + M\rho \tag{38}$$

With the help of equation of continuity, the expressions for *u* and *v* will be

$$u = \exp(\lambda z)\{C_1\cos(\lambda z) + C_2\sin(\lambda z)\} + \exp(-\lambda z)\{C_3\cos(\lambda z) + C_4\sin(\lambda z) - \frac{1}{M\rho}\frac{\partial P}{\partial y} \tag{39}$$

$$v = \exp(\lambda z)\{d_1\cos(\lambda z) + d_2\sin(\lambda z)\} + \exp(-\lambda z)\{d_3\cos(\lambda z) + d_4\sin(\lambda z) + \frac{1}{M\rho}\frac{\partial P}{\partial x} \tag{40}$$

Where

$$\lambda = \sqrt{\frac{M\rho}{2\mu}} \tag{41}$$

$$C_1 = \frac{1}{D}\{a_1\ \sinh(\lambda h)\cos(\lambda h) - a_2\cosh(\lambda h)\} \tag{42}$$

$$C_2 = \frac{1}{D}\{a_1\ \cosh(\lambda h)\sin(\lambda h) - a_2\cos(\lambda h)\sinh(\lambda h)\} \tag{43}$$

$$C_3 = U_1 + \frac{1}{M\rho}\frac{\partial P}{\partial y} - C_1 \tag{44}$$

$$C_4 = -\frac{1}{M\rho}\frac{\partial P}{\partial x} + C_2 \tag{45}$$

$$a_1 = \frac{1}{2M\rho}\left[M\rho U_2 + \frac{\partial P}{\partial y} + \exp(-\lambda h)\left\{\frac{\partial P}{\partial x}\sin(\lambda h) - \left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\cos(\lambda h)\right\}\right] \tag{46}$$

$$a_2 = \frac{1}{2M\rho}\left[\frac{\partial P}{\partial x} - \exp(-\lambda h)\left\{\frac{\partial P}{\partial x}\cos(\lambda h) + \left(M\rho U_1 + \frac{\partial P}{\partial y}\right)\sin(\lambda h)\right\}\right] \tag{47}$$

$$D = cos^2(\lambda h)sinh^2(\lambda h) + cosh^2(\lambda h)sin^2(\lambda h) \tag{48}$$

$$d_1 = -\frac{1}{D}\{a_1\ \sin(\lambda h)\cosh(\lambda h) + a_2\ \sinh(\lambda h)\cos(\lambda h) \tag{49}$$

$$d_2 = \frac{1}{D}\{a_1\ \sinh(\lambda h)\cos(\lambda h) - a_2\ \sin(\lambda h)\cosh(\lambda h) \tag{50}$$

$$d_3 = -\left(\frac{1}{M\rho}\frac{\partial P}{\partial x} + d_1\right) \tag{51}$$

$$d_4 = -\left(\frac{1}{M\rho}\frac{\partial P}{\partial y} + U_1\right) + d_2 \tag{52}$$

On taking the expansion of *u* and *v* in the positive integral powers of *M*, *Banerjee et al.* [1, 3] have written the expressions as follows:

$$u = \left\{\frac{1}{2\mu}\frac{\partial P}{\partial x}z(z-h) + \frac{h-z}{h}U_1 + \frac{z}{h}U_2\right\} - \left\{\frac{\rho}{24\mu^2}\frac{\partial P}{\partial y}z(z^3 - 2z^2h + h^3)\right\}M + \cdots \tag{53}$$

$$v \quad = \frac{1}{2\mu}\frac{\partial P}{\partial y}z(z-h) + \begin{bmatrix} \frac{\rho}{24\mu^2}\frac{\partial P}{\partial y}z(z^3 - 2z^2h + h^3) \\ + \frac{\rho}{6\mu h}z\begin{Bmatrix} (U_1 - U_2)z^2 \\ +3U_1zh - (2U_1 + U_2)h^2 \end{Bmatrix} \end{bmatrix} M + \cdots \tag{54}$$

Now the expansions of above expressions give the values obtained by *Banerjee et al.* [3], for *u* and *v* for small rotation.

By substituting the values of *u* and *v* in the equation of continuity, they have given

$$\frac{\partial(\rho w)}{\partial z} = -\frac{\partial}{\partial x}\left[-\frac{1}{M}\frac{\partial P}{\partial y} + \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)cos(\lambda z) - \frac{\partial P}{\partial x}\sin(\lambda z)\right\} + 2\rho\{C_1\cos(\lambda z)\sinh(\lambda z) + C_2\sin(\lambda z)\cosh(\lambda z)\}\right]$$

$$-\frac{\partial}{\partial y}\left[\frac{1}{M}\frac{\partial P}{\partial x} - \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)sin(\lambda z) - \frac{\partial P}{\partial x}\cos(\lambda z)\right\} + 2\rho\{d_1\cos(\lambda z)\sinh(\lambda z) + d_2\sin(\lambda z)\cosh(\lambda z)\}\right] \tag{55}$$

Taking integration for *z* by using the conditions that $w=w_0/q_c=w_1$ at $z=0$ and $w=w_h/q_c=w_2$ at $z=h$, we have

$$\rho(w_2 - w_1) = -\int_0^h\frac{\partial}{\partial x}\begin{bmatrix}-\frac{1}{M}\frac{\partial P}{\partial y} + \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)cos(\lambda z) - \frac{\partial P}{\partial x}\sin(\lambda z)\right\} \\ +2\rho\{C_1\cos(\lambda z)\sinh(\lambda z) + C_2\sin(\lambda z)\cosh(\lambda z)\}\end{bmatrix}dz - \int_0^h\frac{\partial}{\partial y}\left[\frac{1}{M}\frac{\partial P}{\partial x} - \frac{\exp(-\lambda z)}{M}\left\{\left(M\rho U_1 + \frac{\partial P}{\partial y}\right)sin(\lambda z) - \frac{\partial P}{\partial x}\cos(\lambda z)\right\} + 2\rho\{d_1\cos(\lambda z)\sinh(\lambda z) + d_2\sin(\lambda z)\cosh(\lambda z)\}\right]dz \tag{56}$$

But *h* is the function of *x* and *y*, then we have

$$\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\}+\frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\}=-\frac{\partial}{\partial x}\left[\frac{\rho}{2}(U_1+U_2)\{h+M\rho\psi_2(h)\}\right]-\frac{\partial}{\partial y}\left[\frac{M\rho^2}{2}(U_1+U_2)\psi_1(h)\right]-\rho(w_2-w_1)+U_2\frac{\partial}{\partial x}(\rho h) \quad (57)$$

Where $\psi_1(h)$ and $\psi_2(h)$ are given by

$$\psi_1(h)=-\frac{1}{\lambda M\rho}\frac{\sinh(\lambda h)-\sin(\lambda h)}{\cosh(\lambda h)+\cos(\lambda h)} \quad (58)$$

$$\psi_2(h)=-\frac{h}{M\rho}+\frac{1}{\lambda M\rho}\frac{\sinh(\lambda h)+\sin(\lambda h)}{\cosh(\lambda h)+\cos(\lambda h)} \quad (59)$$

The equation (40) was the extended generalized Reynolds equation. In most of the cases the bearing is stationary and the runner in the thrust bearing and shaft in the journal bearing are moving, so that

$$U_1=U,\ U_2=0 \quad (60)$$

Then the extended generalized Reynolds equation becomes

$$\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\}+\frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\}=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}\{h+M\rho\psi_2(h)\}\right]-\frac{\partial}{\partial y}\left[\frac{M\rho^2U}{2}\psi_1(h)\right]-\rho(w_2-w_1) \quad (61)$$

This is the same for both thrust and journal bearing with $U$ as sliding velocity of either runner or shaft of the bearing. For pure sliding $w_1=w_2$, then we have

$$\frac{\partial}{\partial x}\left\{\rho\psi_1(h)\frac{\partial P}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\rho\psi_1(h)\frac{\partial P}{\partial y}\right\}+\frac{\partial}{\partial x}\left\{\rho\psi_2(h)\frac{\partial P}{\partial y}\right\}-\frac{\partial}{\partial y}\left\{\rho\psi_2(h)\frac{\partial P}{\partial x}\right\}=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}\{h+M\rho\psi_2(h)\}\right]-\frac{\partial}{\partial y}\left[\frac{M\rho^2U}{2}\psi_1(h)\right] \quad (62)$$

The "First order rotatory theory" and "Second order rotatory theory" of Hydrodynamic Lubrication [2, 3] was given by *Banerjee et.al.* [1, 3] on retaining the terms containing up to first and second powers of $M$ [1] respectively, and neglecting higher powers of $M$ [1].

## 6. Conclusions

The Extended Generalized Reynolds Equation in view of first and second order rotatory theory of hydrodynamic lubrication, in ascending powers of rotation number $M$ and by retaining the terms containing up to first and second powers of $M$ and neglecting higher powers of $M$, are shown by equations (63) and (64) respectively as follows:

$$\frac{\partial}{\partial x}\left[-\frac{h^3}{12\mu}\rho\frac{\partial P}{\partial x}\right]+\frac{\partial}{\partial y}\left[-\frac{h^3}{12\mu}\rho\frac{\partial P}{\partial y}\right]+\frac{\partial}{\partial x}\left[-\frac{M\rho^2h^5}{120\mu^2}\frac{\partial P}{\partial y}\right]-\frac{\partial}{\partial y}\left[-\frac{M\rho^2h^5}{120\mu^2}\frac{\partial P}{\partial x}\right]=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}h\right]+\frac{\partial}{\partial y}\left[\frac{M\rho^2U}{2}\frac{h^3}{12\mu}\right]-\rho W^* \quad (63)$$

$$\frac{\partial}{\partial x}\left[-\frac{h^3}{12\mu}\left(1-\frac{17M^2\rho^2h^4}{1680\mu^2}\right)\rho\frac{\partial P}{\partial x}\right]+\frac{\partial}{\partial y}\left[-\frac{h^3}{12\mu}\left(1-\frac{17M^2\rho^2h^4}{1680\mu^2}\right)\rho\frac{\partial P}{\partial y}\right]+\frac{\partial}{\partial x}\left[-\frac{M\rho^2h^5}{120\mu^2}\left(1-\frac{31M^2\rho^2h^4}{3024\mu^2}\right)\frac{\partial P}{\partial y}\right]-\frac{\partial}{\partial y}\left[-\frac{M\rho^2h^5}{120\mu^2}\left(1-\frac{31M^2\rho^2h^4}{3024\mu^2}\right)\frac{\partial P}{\partial x}\right]=-\frac{\partial}{\partial x}\left[\frac{\rho U}{2}\{h-\frac{M^2\rho^2h^5}{120\mu^2}\left(1-\frac{31M^2\rho^2h^4}{3024\mu^2}\right)\}\right]-\frac{\partial}{\partial y}\left[\frac{M\rho^2U}{2}\{-\frac{h^3}{12\mu}\left(1-\frac{17M^2\rho^2h^4}{1680\mu^2}\right)\}\right]-\rho W^* \quad (64)$$


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