International Journal of Applied Research 2016; 2(1): 727-734

ISSN Print: 2394-7500
ISSN Online: 2394-5869
Impact Factor: 5.2
IJAR 2016; 2(1): 727-734
www.allresearchjournal.com
Received: 26-11-2015
Accepted: 30-12-2015

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Interplay between measure theory and topology

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Abstract
Topology and measure theory are very closely related. Both are concerned with spaces equipped with certain algebras of sets (open sets, measurable sets) and classes of functions (continuous functions, measurable functions). Continuous functions (on reasonable spaces) are measurable, and (some) measures can be integrated to define continuous functions. To every topological space X one can associate the Borel σ-algebra, which is the σ-algebra generated by all open sets in X. In this paper we will prove that for locally compact Hausdorff space, ξ be any class of open sets which generates the topology of X, then the σ-ring generated by ξ contains every Baire sets and define a Baire measure on product spaces.

Keywords: Measurable function, Hausdorff Space, Locally Compact Hausdorff Space, Baire set, σ-ring, σ-algebra, compact G_δ set.

Introduction

Preliminaries: Through this paper X denote a Locally Compact Hausdorff Space. Thus X is completely regular. Therefore for every closed subset F of X and x ∉ F and a continuous function f: X → [a,b] such that f(x) = b and f = a on F and $a \leq f \leq b$.

Note: In the Hausdorff space X the following statements are equivalent:
1) Every x ∈ X has a compact neighborhood.
2) For every neighborhood U of x there exist a compact neighborhood V of x s.t. V ⊂ U i.e. every neighborhood contains a compact neighborhood.

3) For every neighborhood U of x there exist a neighborhood V of x s.t. V ⊂ Uand $\bar{V}$ is compact.

Definition: Let f be any real valued function defined on X. Let A = {f ≠ 0}, then $\bar{A}$ is called support of the function f.

Definition: By $\mathcal{L}(x)$ we denote the class of all continuous functions on X which have compact support.

Note: f has compact support iff f is zero outside a compact set.

Proof: Suppose f has compact support. Let A = {f ≠ 0}, then $\bar{A}$ is compact. If x ∉ $\bar{A}$ then x ∉ A ⇒ f(x) =0 ⇒ f = 0 on X-$\bar{A}$ and $\bar{A}$ is compact.

Conversely: Assume that f vanishes outside a compact set. Let K be a compact set s.t. f= 0 on X-K.
Let B = {f ≠ 0} and letx∈ B then f(x) ≠ 0 ⇒ x ∉ X-K 0 ⇒ x ∈ K ⇒ B ⊂ K ⇒ $\bar{B} \subset \bar{K}$ and $\bar{K}$ is compact set ⇒ $\bar{B}$ is a closed subsetof a compact set⇒ $\bar{B}$ is compact set ⇒ f has compact support.

Preposition: By $\mathcal{L}(x)$ we denote the class of all continuous functions on X which have compact support. Let f, g ∈ $\mathcal{L}(x)$ then
1. f ± g ∈ $\mathcal{L}(x)$

2. f $g \in \mathcal{L}(x)$
3. f $\vee\, g \in \mathcal{L}(x)$
4. f $\wedge\, g \in \mathcal{L}(x)$

Proof

1. Obviously $f \pm g$ is continuous. Let K_1 and K_2 be compact sets s.t. f = 0 on X- K_1 and g = 0 on X- K_2. Let K = $K_1 \cup K_2$, then K is compact and let x ∈ X- K ⇒ x ∉ K ⇒ x ∉ $K_1 \cup K_2$ ⇒ x ∉ K_1andx ∉ K_2 ⇒ f(x) = 0 and g(x) = 0⇒ f ± g = 0 ⇒ f ± g = 0 on X-K and K is compact set, Hence proves that f ± g ∈ $\mathcal{L}(x)$.
2. Obvious.
3. Here f $\vee\, g = \frac{f+g+|f-g|}{2}$, write ϕ = f + g and ψ = |f-g|, c = $\frac{1}{2}$ then f $\vee\, g$ = c ($\phi + \psi$), Since f + g ∈ $\mathcal{L}(x)$ we get and f- g ∈ $\mathcal{L}(x)$ ⇒ f + g ∈ $\mathcal{L}(x)$, |f- g| ∈ $\mathcal{L}(x)$ ⇒ $\phi + \psi \in \mathcal{L}(x)$ ⇒ c($\phi + \psi$) ∈ $\mathcal{L}(x)$ ⇒ f$\vee\, g \in \mathcal{L}(x)$.
4. Similarly f $\wedge\, g \in \mathcal{L}(x)$.

Preposition: Let x ∈ X and V be any neighborhood of x, then there exist f ∈ $\mathcal{L}(x)$ such that f(x) = 1, f = 0 on X- V, $0 \le f \le 1$.

Proof: Since X ix locally compact, there exists a neighborhood U of x such that U ⊂ V and $\bar{U}$ is compact, Let f be a real valued continuous function, f: X → [0, 1] such that f(x) = 1 and f = 0 on X – U. Then f = 0 on X – V and f = 0 on X - $\bar{U}$, shows that $f \in \mathcal{L}(x)$, f(x) = 1, f = 0 on X- V,
$0 \le f \le 1$.

Theorem: Let C be any compact set and U be any open superset of C, then there exist $f \in \mathcal{L}(x)$ such, $0 \le f \le 1$, f = 1 on C and f = 0 on X – U.

Proof: Consider any x ∈ C, then x ∈ U, By the above theorem there exist $f_x \in \mathcal{L}(x)$s.t.
$0 \le f_x \le 2$, $f_x(x) = 2$ and $f_x = 0$ on X-U.
Define $U_x = f_x^{-1}$ (1,∞), Then U_x isan open set and,f_x(x) = 2 ⇒ x ∈ U_x.
Show {U_x : x∈ C } is an open cover of C. Since C is compact, there exist a finite sub cover i.e. finitely many elementsx_1,x_2 … …. , x_n such that C ⊂ $\bigcup_{j=1}^{n} x_j$. Defineg = $f_{x_1} + f_{x_2} + \cdots \ldots + f_{x_n}$
Then g ∈ $\mathcal{L}(x)$, g = 0 on X –U, g ≥ 0, g ≥ 1 on C. { Becausex ∈ C⇒ x ∈ U_{x_i} for some i f_{x_i}(x)∈ (1, ∞) ⇒ g> 1}
Define further f = min {1, g}, then f is continuous as minimum of two continuous functions is also continuous.
Suppose g = 0 on X-K where K is compact set. Then f =0 on X- K. Hence f ∈ $\mathcal{L}(x)$ on X- V,
g = 0 ⇒ f = 0 on X- U
On C, g > 1, therefore f =1 on C and $0 \le f \le 1$. This completes the proof.

Definition: A countable intersection of open sets is called a G_δ set.

Remark

1. Every open set is a G_δ set.
2. Countable intersection of G_δ sets is a G_δ set.
3. Finite union of G_δ sets is aG_δ set.
4. If A is any $G_\delta$set then there exist decreasing sequence {A_n} of$G_\delta$sets such that A_n → A.

Proof:1, 2, 3 are obvious.

4. Let A be a $G_\delta$set then A = $\bigcap_{1}^{\infty} G_n$ where G_n are open sets, Define $A_n = \bigcap_{j=1}^{n} G_j$, then A_n is a decreasing sequence of G_δ sets, therefore $\lim_{n\to\infty} A_n = \bigcap_{1}^{\infty} A_n = \bigcap_{1}^{\infty} G_n$= A.

Definition: A countable union of closed sets is called F_σ sets.

Remark

1. Every closed set is a F_σ set.
2. Countable union of F_σ sets is a F_σ set.
3. Finite intersection of F_σ sets is aF_σ set.
4. If B is any $F_\sigma$set then there exist monotone increasing sequence {B_n} of$F_\sigma$sets such that B_n → B.

Proof: **4.** Let B = $\bigcup_{1}^{\infty} F_n$ where F_n are closed sets, Define $B_n = \bigcup_{j=1}^{n} F_j$, then B_n is an increasing sequence of $F_\sigma$sets, therefore $\lim_{n\to\infty} B_n = \bigcup_{1}^{\infty} B_n = \bigcup_{1}^{\infty} G_n$= B.

Preposition: Let f : X → $\mathcal{R}$ be continuous and $\alpha \in \mathcal{R}$. Suppose A = {f ≥ α}, B= {f ≤ α }, C ={f= α },then

1. A, B, C are closed G_δ sets.
2. If f has compact support i.e. f ∈ $\mathcal{L}(x)$ and $\alpha > 0$ then A and C compact G_δ sets.
3. If f has compact support i.e. f ∈ $\mathcal{L}(x)$ and $\alpha < 0$ then B and C compact G_δ sets.

Proof:

1. Since f is continuous, it follows immediately that A, B and C are closed sets,
   A = $\{f \geq \alpha\} = \{x \in X : f(x) \geq \alpha\} = f^{-1}[\alpha, \infty]$ ⇒ A is closed.
   Similarly B is closed and C = A ∩B ⇒ C is closed.
   A = $\bigcap_{n=1}^{\infty} \{f > \alpha - \frac{1}{n}\}$ the continuityof fgives that $\{ f > \alpha - \frac{1}{n}\} = f^{-1}(\alpha - \frac{1}{n}, \infty)$ is an open set for all n. Hence A is a G_δ set.
   Similarly we can say that B = $\bigcap_{n=1}^{\infty} \{f < \alpha + \frac{1}{n}\}$ which shows that B is a G_δ set.
   And C= A ∩B ⇒ C is also a G_δ set.
2. Suppose f ∈ $\mathcal{L}(x)$ and $\alpha > 0$, let D be a compact set such that f = 0 on X-D.
   If x ∈ A then f(x) ≥ α ⇒ f(x) > 0 ⇒ f(x) ≠ 0 ⇒ x ∉ X- D ⇒ x∈ D ⇒ A ⊂ D
   Thus A is a closed set which contains in a compact set D. Hence A is compact.
   Similarly C ⊂ D ⇒ C is a compact set.
   We already proved that A and C are G_δ sets.
3. Suppose f ∈ $\mathcal{L}(x)$ and $\alpha < 0$, let K be a compact set such that f = 0 on X-K.
   If y ∈ B then f(y) ≤ α ⇒ f(y) < 0 ⇒ f(y) ≠ 0 ⇒ y ∉ X- K ⇒ y∈ K ⇒ B ⊂ K
   Thus B is a closed subset of a compact set K. Hence B is compact.
   Similarly C ⊂ K ⇒ C is a compact set.
   We already proved that B and C are G_δ sets.
   Hence the proof.

Definition: *If* $\mathcal{F}$ be any class of subsets of X such that
1. $\mathcal{F}$ is closed for differences i.e. A, B ∈ $\mathcal{F}$ ⇒ A-B ∈ $\mathcal{F}$.
2. $\mathcal{F}$ is closed for countable unions.

Then $\mathcal{F}$ is called a $\sigma -$ ring on X.

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Definition: *If* $\mathcal{F}$ be any class of subsets of X such that
1. If A, B ∈ $\mathcal{F}$ *then* A-B is a finite union of members of $\mathcal{F}$.
2. $\mathcal{F}$ is closed for finite intersections.

Then $\mathcal{F}$ is called a semi − ring on X.

Remark

(1) Every σ- ring is a ring and every ring is a semi ring.
(2) A ring is closed for symmetric differences. i.e. If R is a ring and A and B∈ R then $A \Delta B = (A - B) \cup (B - A) \in R$
(3) A ring is closed for finite intersections.
   i.e. If R be a ring and A,B ∈ R then $A \cap B = (A \cup B) - (A \Delta B) \Rightarrow A \cap B \in R$.
(4) A σ- ring is closed for countable intersections.
   Let S be any σ- ring and (A_n) be any sequence of members of S then
   $\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n)$. If x ∈ A_n then x ∈ A_n ∀ n⇒ x ∈ A_1 and x ∉ $A_1 - A_n$ ∀ n
   ⇒x∈ A_1 and x ∉ $\bigcup_{n=1}^{\infty} (A_1 - A_n)$ ⇒ x∈ $A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n)$.
   Hence L.H.S. ⊆ R.H.S……..(1)
   Now suppose that y ∈ $A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n)$, then y ∈ A_1 and y∉ $\bigcup_{n=1}^{\infty} (A_1 - A_n)$
   ⇒y∈ A_1 and y ∉ $(A_1 - A_n)$ ∀ n⇒ y ∈ A_1 and y ∈ A_n ∀ n⇒ y ∈ $\bigcap_{n=1}^{\infty} A_n$
   ⇒ R.H.S. ⊆ L.H.S. ……(2)
   From (1) and (2) we prove that $\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n) \in S$.

Example: Let S be a class of countable subsets of R.
If A and B ∈ S then A, B both are countable ⇒ A-B are countable ⇒ A-B ∈ S.
Let (A_n) be any sequence of countable sets then A_n is countable ∀ n ⇒ $\bigcup_{n=1}^{\infty} (A_n)$ is countable

$\Rightarrow \bigcup_{n=1}^{\infty}(A_n) \in$ S, shows that S is a σ- ring. Thus S is not a $\sigma-$ algebra because
If A be the set of all rational numbers then A is countable and A^cwhich is the set of irrational numbers is uncountable. Therefore A ∈ S and $A^c \notin$ S, Which shows that S is not a $\sigma-$ algebra.

Preposition: If $\{R_\alpha\}$be the family of rings on X, then $\bigcap_\alpha R_\alpha$ is also a ring on X.

Definition: Let ξ be any class of subsets of X and $\{R_\alpha\}$be the class of all rings on X which contains ξ. Let R = $\bigcap_\alpha R_\alpha$, then R is a ring containing $\xi$and is the smallest ring containing ξ called the ring generated by ξ and we denote it by R(ξ).

Preposition: Let R(ξ) consists of sets which are covered by any finite union of members of ξ.

Preposition: Let $\{S_\alpha\}$ be any class of σ-rings on X, then $\bigcap_\alpha S_\alpha$is also a σ-ring on X.

Definition: Let ξ be any class of subsets of X and $\{s_\alpha\}$ be the family of all σ-rings which contains ξ. Let S = $\bigcap_\alpha S_\alpha$ then S is a σ-ring containing ξ. This ring is called the σ-ring generated by ξ and is denoted by $\mathfrak{S}(\xi)$.

Preposition:$\mathfrak{S}(\xi)$ consists of all sets which can be covered by countable unions of members of ξ.

Definition: Let $\mathcal{M}$ be any class of sets of X such that $\mathcal{M}$ is closed for limits of monotone sequences of its members. That if (A_n) be any monotone increasing sequence of members of $\mathcal{M}$ then $\bigcup_1^\infty A_n \in \mathcal{M}$ and if if (A_n) be any monotone decreasing sequence of members of $\mathcal{M}$then $\bigcap_1^\infty A_n \in \mathcal{M}$, Then $\mathcal{M}$ is called a Monotone class.

Note
(1)Every σ-ring is a monotone class.
(2)A monotone class $\mathcal{M}$ is a σ-ring iff $\mathcal{M}$ is a ring.
If $\mathcal{M}$ is a σ-ring, then $\mathcal{M}$ is a ring (Obvious).

Conversely: Suppose that $\mathcal{M}$ is a ring,(A_n) be any sequence of members of $\mathcal{M}$.
Define $B_n = \bigcup_{i=1}^{n} A_i$ then $B_n \in \mathcal{M}\ \forall$ n. Let A= $\bigcup_1^\infty A_n$ then A= $\bigcup_1^\infty B_n \Rightarrow (B_n)\uparrow$A⇒ A ∈ $\mathcal{M}$ as $\mathcal{M}$ is a monotone class ⇒ $\mathcal{M}$ is a $\sigma-$ ring.

Preposition: Intersection of monotone classes is a monotone class.

Definition: Let ξ be any class of subsets of X, let $\{\mathcal{M}_\alpha\}$ be the class of all monotone classes which contains ξ and $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$ then $\mathcal{M}$ is a monotone class, $\mathcal{M}$ contains ξ and $\mathcal{M}$ is the smallest monotone class containing ξ. This $\mathcal{M}$ is called the monotone class generated by ξ and we denote it by $\mathcal{M}(\xi)$.

Remark: If R is ring then $\mathfrak{S}(R)$= $\mathcal{M}(R)$.

Corollary: If R is a ring, $\mathcal{M}$ is a monotone class and R ⊂ $\mathcal{M}$ then $(R) \subset \mathcal{M}$. Thus $\mathfrak{S}(R)$= $\mathcal{M}(R) \subset \mathcal{M}$.

Proposition: Let K be any compact G_δ set, then there exist a monotone decreasing sequence (f_n) of members of $\mathcal{L}(X)$ such that $f_n \to C_k$, where C_k is the characteristic function of K.

Proof: Since K is a G_δ set, there exist a monotone decreasing sequence $\{U_n\}$ of open sets such that $U_n \downarrow$ K, then K = $\bigcap_{n=1}^{\infty} U_n$. Consider any n, a natural number then K ⊂ U_n, By the earlier result we can find a function $g_n \in \mathcal{L}(X)$ such that $g_n = 1$ on K and $g_n = 0$ on X- U_n
$\Rightarrow 0 \le g_n \le 1$.
Definef_n = min $\{g_1, g_2, \dots \dots, g_n\}$ then $f_n \in \mathcal{L}(X)$ and$f_1 \ge f_2 \ge f_3 \ge \cdots$ ... and $f_n = 1$ on K and $f_n = 0$ on X- U_n ∀n.
Let x ∈ X, suppose x ∈ K then f_n(x) = 1 ∀ n ⇒ $\lim_{n\to\infty} f_n$(x)= 1 = C_k(x)…………(1)
Suppose x ∉ K, then there exist m such that x ∉ U_m,Let n ≥ m, then$U_n \subset U_m$
⇒x ∉ U_n ⇒ f_n(x) = 0 ∀ n ≥ m,hence $\lim_{n\to\infty} f_n$(x)= 0 = C_k(x)………………..(2)
From (1) and (2) we prove that $f_n \to C_k$.

Definition: Let X be any locally compact Hausdorff space, then the σ-ring generated by compact G_δ sets is called the σ-ring of Baire Sets or Baire σ-ring. We shall denote this σ-ring by Ω and the members of Ω are called the Baire Sets.

Definition: A topological space X is a $T_q -$ space (Hausdorff) iff x, y are two distinct points of X, then there exist two disjoint open sets one containing x but not y and the other containing y but not x.

Theorem (Baire Sandwhich Theorem): Suppose C is compact and U is an open set and C ⊂ U, then there exist Baire Sets V and D such that

(1) C ⊂ V ⊂ D ⊂ U
(2) V is open, D is compact G_δ set.
(3) V is a countable union of compact G_δ sets i.e. V is σ-compact.

Proof

(1) Let $f \in \mathcal{L}(X)$ such that $f = 1$ on C and $f = 0$ on X-U, then Define V = $\{ f > \frac{1}{2}\}$ and D = $\{ f \geq \frac{1}{2}\}$ then C ⊂ V ⊂ D ⊂ U.
(2) As $(\frac{1}{2}, \infty)$ is open and f is continuous it is obvious that V =$f^{-1}(\frac{1}{2}, \infty)$ is open.
As $f \in \mathcal{L}(X)$, it follows that D is compact G_δ set.
(3) V = $\bigcup_{n=1}^{\infty} \{ f \geq \frac{1}{2} + \frac{1}{n} \} = \bigcup_{n=1}^{\infty} D_n$ where $D_n = \{ f \geq \frac{1}{2} + \frac{1}{n} \}$

By the argument given above D_n is compact G_δ set, and then V is a countable union of compact G_δ sets. As D is compact G_δ set, D ∈ Ω and V = $\bigcup_{n=1}^{\infty} D_n, D_n \in \Omega \forall$ n, it follows that V ∈ Ω. Thus V and D are both Baire Sets.

Corollary

(1) Open Baire sets form a base for X.
(2) Open Baire neighborhoods of x form a local base at x.
(3) Compact G_δ neighbourhoods of x form a local base for x.

Definition

(1) $\mathfrak{B}$ is a base for X if G ⊂ X, G is open ⇒ G $= U^{P}_{P\in \mathfrak{B}}$. If x ∈ G ⇒ there exist $P \in \mathfrak{B}$ s.t. x ∈ P, P ⊂ G, $P \in \mathfrak{B}$.
(2) G is open, x ∈ G ⇒ there exist $P \in \mathfrak{B}$ s.t. x ∈ P ⊂ G.

Definition: Let S be a σ-ring on X and f be any extended real-valued function defined on X.
If $\{f > \alpha\}$ is locally measurable for every∈ R, then f is said to be S-measurable.
[i.e. $\{f > \alpha\} \cap E \in S \; \forall \; E \in S\}$

Definition: If f is Ω measurable function then f is called a Baire function, where Ω is the σ-ring of Baire sets.

Preposition: Every f ∈ $\mathcal{L}(X)$ is a Baire function.

Proof: Let $\alpha \in$ R. Suppose first that f ≥ 0, then $\{f \geq \alpha\} = X$ if $\alpha \leq 0$
If $\alpha > 0$ then $\{f \geq \alpha\}$ is a compact G_δ set.
Shows that f *is* Ω-measurable function.Assume that f is any general function, f ∈ $\mathcal{L}(X)$we see that $f^+ \in \mathcal{L}(X)$ and $f^- \in \mathcal{L}(X)$.
By what has been proved above we get f^+ and f^- both are Ω-measurable functions ⇒ $f^+ - f^-$ is measurable
⇒ f is Ω-measurable function i.e. f is a Baire function.

Corollary: Let S be a σ-ring on X, then the following statements are equivalent:

(1) S contains every compact G_δ set.
(2) S contains every Baire set.
(3) Every function in $\mathcal{L}(X)$is S-measurable.

Proof: It is obvious that (1) ⇒ (2)
To show (2) ⇒ (3) Suppose that (2) holds, let f ∈ $\mathcal{L}(X)$ and $\alpha \in$ R by preposition f is a Baire function ⇒$\{ f > \alpha \}$ is a Baire set.
⇒ $\{f > \alpha\} \in S$ because S contains every Baire set by supposition and this shows that f is S-measurable.
To show (3) ⇒ (1)
Assume that (3) holds, Let K be any compact G_δ set, as proved earlier there exist a sequence $\{f_n\}$ of members of $\mathcal{L}(X)$s.t. $\{f_n\} \downarrow C_k$. As (3) holds, f_n is a S-measurable for all n, ⇒ C_k is a S-measurable ⇒ K is measurable ⇒ K ∈ S. Hence the proof.

Definition: Let X be L.C.H. space, Ω be a σ-ring of Baire sets. Let μ be any measure on Ω s.t. $\mu(K) < \infty$ for every compact G_δ set K. Then μ is called a Baire measure on X.

Preposition: Every Baire measure is σ-finite.

Proof: Let X be L.C.H. space, Ω be a σ-ring of Baire sets and μ be any measure on Ω.

Let $\mathfrak{C}$ denote the class of compact G_δ sets, then $\Omega = \mathfrak{S}(\mathfrak{C}) \Rightarrow$ every member of Ω is contained in a countable union of members of $\mathfrak{C}$. Let S $\in \Omega$ then S $\subset \bigcup_{n=1}^{\infty} k_n$, where k_n are compact G_δ sets. By definition of Baire measure $\mu(K)<\infty$ for every n, which shows that every Baire set is contained in a countable union of sets of finite measure. Hence μ is σ-finite.

Lemma: Let X be any set $\mathfrak{C}$ be the class of subsets of X such that $\phi \in \mathfrak{C}$, A ∪ B, A ∩ B ∈ $\mathfrak{C}$ whenever A, B ∈ $\mathfrak{C}$.Let R be the ring generated by $\mathfrak{C}$, then every member of R is of the type $\bigcup_{n=1}^{n}(A_i - B_i)$, whenever A_i, $B_i \in \mathfrak{C}$ and $A_i \supset B_i$, $A_i - B_i$ are disjoint and n be a natural number.

Proof: Let R_0 denote the class of sets of the type $\bigcup_{i=1}^{n}(A_i - B_i)$…………..(*)
Since R is a ring containing $\mathfrak{C}$, it follows that $R_0 \subset$ R.
We want to show that R ⊂ R_0. For this we have to prove that R_0 is a ring.
Let A and B ∈ $\mathfrak{C}$ then A-B = A- (A∩ B)
Taking A_1= Aand B_1= A∩ B and n = 1 we find that A-B is a set of the type (*)
hence A-B ∈ R_0.
Thus R_0 is closed for differences of members of $\mathfrak{C}$……………..…(1)
It is easy to observe that R_0 is closed for finite disjoint unions….…(2)
Let E,F ∈ R_0 and E = $\bigcup_{i=1}^{n}(A_i - B_i)$, F= $\bigcup_{j=1}^{m}(C_j - D_j)$ where $A_i \supset B_i$ and A_i, $B_i \in \mathfrak{C}, A_i - B_i$ are disjoint for 1≤ i ≤ n and same is holds for $C_j \& D_j$.
Then E∩ F = $[\bigcup_{i=1}^{n}(A_i - B_i)] \cap [\bigcup_{j=1}^{m}(C_j - D_j)] = \bigcup_{i=1}^{n}[(A_i - B_i) \cap \{\bigcup_{j=1}^{m}(C_j - D_j)\}]$
$= \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}[(A_i - B_i) \cap (C_j - D_j)] = \bigcup_{ij}\{((A_i \cap C_j) - (B_i \cap D_j)].$
Since $\mathfrak{C}$ is closed for finite unions and finite intersections it follows that E ∩ F is a set of the type (*). Hence E ∩ F $\in R_0$ this shows that R_0 is closed for finite intersections ……………………(3)
Let $G_i = \bigcup_{j=1}^{m}[(A_i - B_i) - (C_j - D_j)]$,then E-F = $\bigcup_{i=1}^{n} G_i$,
in general (P-Q)-(R-S) = [P-(Q∪ R) ∪ [(P ∩ S) − Q]
If R⊃ S then the two sets on right hand side is disjoint
Therefore $G_i = \bigcup_{j=1}^{m}[\{A_i - (B_i \cup C_j)\} \cup \{(A_i \cap D_j) - B_i\}]$ where $A_i, B_i, C_j, D_j \in \mathfrak{C}$ *and* $A_i, B_i \cup C_j$,
$A_i \cap D_j$, $B_i$are all members of $\mathfrak{C}$. By (1) $A_i - (B_i \cup C_j) \in R_0 \Rightarrow G_i$ is a finite disjointunion of members of $R_0 \Rightarrow \bigcup_{i=1}^{n} G_i \in R_0$ by (2)
Hence E-F ∈ R_0, Shows that R_0 is closed for differences………….(4)
Now E∪ F = (E − F) ∪ (F − E) ∪ (E ∩ F), E-F, F-E∈ R_0 . ……..By (4),
Also E ∩ F ∈ R_0 by (3)
Thus E-F, F-E, E ∩ F are disjoint members of R_0. Appealing to (2) we see that E ∪ F $\in R_0$
Which shows that R_0 is closed for finite unions……………………... (5)

From (4) and (5) it follows that R_0 is a ring and that end the proof.

Theorem: Let $\mathfrak{C}$ be the class of subsets of locally compact hausdorff space X, S be the σ-ring generated by $\mathfrak{C}$.Let μ_1, μ_2 be any two measures on S such that (1) $\mu_1(c) = \mu_2(c)\ \forall$ c ∈ $\mathfrak{C}$. (2) μ_1, μ_2 are finite on $\mathfrak{C}$. Then $\mu_1 = \mu_2$ on S.

Proof: Let R be the ring generated by $\mathfrak{C}$. Let s ∈ R, then s = $\bigcup_{i=1}^{n}(A_i - B_i)$ where A_i, $B_i \in \mathfrak{C}$ *and* $A_i \supset B_i$and $A_i - B_i$ are disjoint for 1 ≤ i ≤ n.
This implies that $\mu_1(s) = \sum_{i=1}^{n} \mu_1 (A_i - B_i) = \sum_{i=1}^{n}[\mu_1 (A_i) - \mu_1(B_i)]$ {Becauseμ_1 is finite on $\mathfrak{C}$}
$= \sum_{i=1}^{n}[\mu_2 (A_i) - \mu_2(B_i)]$ { Because $\mu_1(c) = \mu_2(c)\ \forall$ c ∈ $\mathfrak{C}$}
$= \sum_{i=1}^{n} \mu_2 (A_i - B_i) = \mu_2[\bigcup_{i=1}^{n}(A_i - B_i)] = \mu_2(s)$
Thus $\mu_1(s) = \mu_2(s)\ \forall$ s ∈ R, As μ_1 *and* μ_2 are finite on R, then by Carthathedeory's Extension Theorem we conclude that $\mu_1 = \mu_2$ on $\mathfrak{S}(R)$ = S ⇒ $\mu_1 = \mu_2$ on S.

Corollary: Let Ω be the $\sigma -$ ring of Baire sets of X and μ_1, μ_2 be Baire measures such that $\mu_1(K) = \mu_2(K)$ for every compact G_δ set K. Then $\mu_1 = \mu_2$ on Ω.
That is a Baire measure is uniquely determined by its values on compact G_δ sets.

Proof: Let $\mathfrak{C}$ be the class of compact G_δ sets, then $\Omega = \mathfrak{S}(\mathfrak{C})$, Since $\mu_1 = \mu_2$ on $\mathfrak{C}$ and they are finite on $\mathfrak{C}$ itfollows from the theorem that $\mu_1 = \mu_2$ on Ω.

Theorem: Let X be a locally compact hausdorff space, ξ be any class of open sets which generates the topology of X. Then the σ-ring generated by ξ contains every Baire sets.

Proof: Let $\mathfrak{B}$ denote the class of finite intersections of ξ. Then $\mathfrak{B}$ is a base for X.
Let C be any compact set, U be any open set and C ⊂ U. Since $\mathfrak{B}$ is a base for X we can write
U = $\bigcup_{\alpha} V_\alpha$ where $V_\alpha \in \mathfrak{B} \Rightarrow$ C ⊂ $\bigcup_{\alpha} V_\alpha$. As C is compact set there exist finitely many
$V_1, V_2, \ldots, V_n$ such that C ⊂ $\bigcup_{j=1}^{n} V_j$.
Define F = $\bigcup_{j=1}^{n} V_j$, then F ∈ S and C ⊂ F ⊂ U. Thus for every compact set C and for every open set U with C ⊂ U, we can find a set F∈ S such that C ⊂ F ⊂ U……………(*)
Let K be any compact G_δ set, then K = $\bigcap_{n=1}^{\infty} U_n$ where U_n is open for all n.
This gives K ⊂ $U_n \forall$ n.
From (*) we can find $F_n \in$ S such that K ⊂ $F_n \subset U_n \Rightarrow$ K = $\bigcap_{n=1}^{\infty} F_n$ where $F_n \in$ S ⇒ K ∈ S.
Let $\mathfrak{C}$ be the class of compact G_δ sets then we have $\mathfrak{C}$ ⊂ S, Hence $\mathfrak{S}(\mathfrak{C})$ ⊂ S ⇒ Every Baire set belongs to S.

Theorem: Let $\xi \subset$ P(X)the power set of X, $\zeta \subset$ P(Y) then $\mathfrak{S}(\xi \times \zeta) = \mathfrak{S}(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta))$.

Proof: Since $\xi \in \mathfrak{S}(\xi)$ *and* $\zeta \in \mathfrak{S}(\zeta)$ it is clear that $\xi \times \zeta \subset \mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)$
$\Rightarrow \xi \times \zeta \subset \mathfrak{S}\big(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)\big) \Rightarrow \mathfrak{S}(\xi \times \zeta) \subset \mathfrak{S}\big(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)\big)$ ………… (1)
To show that $\mathfrak{S}(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)) \subset \mathfrak{S}(\xi \times \zeta)$
Let E ∈ ξ, Define D = { T ⊂ Y/E × T ∈ $\mathfrak{S}(\xi \times \zeta)$}
Let(T_n) be any sequence of members of D.
⇒ E × $T_n \in \mathfrak{S}(\xi \times \zeta)$ for all n⇒$\bigcup_{n=1}^{\infty}$(E × T_n) ∈ $\mathfrak{S}(\xi \times \zeta) \Rightarrow$ E ×$\bigcup_{n=1}^{\infty}(T_n)$ ∈ $\mathfrak{S}(\xi \times \zeta) \Rightarrow \bigcup_{n=1}^{\infty}(T_n)$ ∈ D. I.e. D is closed for countable unions.
E ×$(T_1 - T_2)$ = (E × T_1) − (E × T_2) ⇒ E ×$(T_1 - T_2) \in \mathfrak{S}(\xi \times \zeta) \Rightarrow T_1 - T_2 \in$ D,

Hence D is closed for differences and hence D is a σ-ring.
Let F∈ ζ then E × $F \in \xi \times \zeta \Rightarrow$ E × $F \in \mathfrak{S}(\xi \times \zeta) \Rightarrow$ F ∈ D ⇒ $\zeta \subset$ D ⇒ $\mathfrak{S}(\zeta) \subset$ D.
If T∈ $\mathfrak{S}(\zeta)$ then T ∈ D ⇒ E × $T \in \mathfrak{S}(\xi \times \zeta)$……………………………(a)
i.e. E × $T \in \mathfrak{S}(\xi \times \zeta) \forall$ E ∈ ξ and T∈ $\mathfrak{S}(\zeta)$

Now consider F∈ $\mathfrak{S}(\zeta)$, Define D^*= {S⊂ X / S × $F \in \mathfrak{S}(\xi \times \zeta)$}
As above we can show that D^* is a σ-ring and ⊂ D^*, this gives that $\mathfrak{S}(\xi) \subset D^*$.
Let S ∈ $\mathfrak{S}(\xi)$ thenS ∈ D^* ⇒S × $F \in \mathfrak{S}(\xi \times \zeta)$.
Thus S ∈ $\mathfrak{S}(\xi)$, F∈ $\mathfrak{S}(\zeta)$ thenS × $F \in \mathfrak{S}(\xi \times \zeta)$……………………..(b)
Therefore from (a) and (b) we have $\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta) \subset \mathfrak{S}(\xi \times \zeta)$
$\Rightarrow \mathfrak{S}(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)) \subset \mathfrak{S}(\xi \times \zeta)$ …………………………………(2)

Hence from (1) and (2) we have$\mathfrak{S}(\mathfrak{S}(\xi) \times \mathfrak{S}(\zeta)) = \mathfrak{S}(\xi \times \zeta)$.

Theorem: Let X *an d Y* beL. C. H. spaces and $\Omega(X), \Omega(Y), \Omega(X \times Y)$denote the σ-rings of Baire sets of X, Y, and X× Y respectively then $\Omega(X \times Y) = \mathfrak{S}(\Omega(X) \times \Omega(Y))$.

Proof: Let ξ, ζ, g denote the classes of compact G_δ sets of X, Y and $X \times Y$ respectively. Consider any C ∈ $\xi, D \in \zeta$ then there exist decreasing sequences of open sets (U_n) of X and (V_n) of Y s.t. $(U_n)\downarrow C$ and $(V_n)\downarrow D \Rightarrow (U_n \times V_n)\downarrow C \times D$.
As $U_n \times V_n$ is an open set of $X \times Y$, it follows that $C \times D$ is a G_δ set of X×Y. By Tychnoff's Theorem that $C \times D$ is a compact set of X×Y. Thus $C \times D$ is a *compact* G_δ set of X×Y i.e. $C \times D \in g$. Shows that $\xi \times \zeta \subset g$.
$\Rightarrow \mathfrak{S}(\xi \times \zeta) \subset \mathfrak{S}(g) \Rightarrow \mathfrak{S}(\Omega(X) \times \Omega(Y)) \subset \Omega(X \times Y)$…………..(1)
Let ξ^*={ G× P / G is a open set of X, P is open set of Y}. Consider G× $P \in \xi^*$, and (x,y) ∈ G× P thenx ∈ G and y ∈ P, By Baire Sandwich Theorem there exist an open Baire set U and V of X and Y respectively. x ∈ U ⊂ G, y ∈ V ⊂ P ⇒(x,y) ∈ U × V ⊂ G× P,

Shows that $\bar{\xi}$= { U × $\frac{V}{}$ U and V are open Baire sets of X and Y respectively} generates the topology of $X \times Y$. Hence $\mathfrak{S}(\xi) \subset$ $\Omega(X \times Y) \Rightarrow \Omega(X \times Y) \subset \mathfrak{S}\left(\bar{\xi}\right)$ But $\bar{\xi} = \Omega(X) \times \Omega(Y)$
$\Rightarrow \mathfrak{S}\left(\bar{\xi}\right) \subset \mathfrak{S}((\Omega(X) \times \Omega(Y)) \Rightarrow \Omega(X \times Y) \subset \mathfrak{S}((\Omega(X) \times \Omega(Y))$
Hence we get $\Omega(X \times Y) = \mathfrak{S}((\Omega(X) \times \Omega(Y))$. Proved.

Theorem: *Let* $\Omega(X), \Omega(Y), \Omega(X \times Y)$ denote the σ-rings of Baire sets of X, Y, and $X \times Y$ respectively, μ *and* v be Baire measures on X and Y then $\mu \times v$ is a Baire measure on $X \times Y$.

Proof: Let $\pi = \mu \times v$ then the domain of π is $\mathfrak{S}((\Omega(X) \times \Omega(Y))$.
But from the above result we have $\mathfrak{S}((\Omega(X) \times \Omega(Y)) = \Omega(X \times Y)$, hence the domain of π is $\Omega(X \times Y)$.
Let K be any compact G_δ set of $X \times Y$. Consider (x,y) ∈ K then (x,y) ∈ $X \times Y \Rightarrow x \in X, y \in Y$

By local compactness x has an open neighborhood U_x s.t. $\bar{U_x}$ is compact, similarly has a open neighborhood V_y s.t. $\bar{V_y}$ is compact. Clearly (x,y) ∈ $U_x \times V_y$ shows that
$\{U_x \times V_y$ / (x,y) ∈ K$\}$ is an open covering for K. Since K is compact there exist finitly many sets
$U_1 \times V_1, U_2 \times V_2, \ldots\ldots\ldots, U_n \times V_n$ s.t. $K \subset \bigcup_{i=1}^{n} (U_i \times V_i)$.

Since $\bar{U_i}$ is compact and $\bar{U_i} \subset X$, By Baire Sandwich Theorem there exist a compact G_δ set G_i of X s.t. $\bar{U_i} \subset G_i \subset X$, Similarly there exist a compact G_δ set D_i of Y s.t. $\bar{V_i} \subset D_i \subset Y$.
This implies $K \subset \bigcup_{i=1}^{n} (U_i \times V_i) \subset \bigcup_{i=1}^{n} (\bar{U_i} \times \bar{V_i}) \subset \bigcup_{i=1}^{n} (C_i \times D_i)$.
Then $\pi(K) \leq \sum_{i=1}^{n} \pi(C_i \times D_i) = \sum_{i=1}^{n} (\mu \times v)(C_i \times D_i) = \sum_{i=1}^{n} \mu(C_i)\, v\, (D_i) < \infty$ because μ & v are finite on compact G_δ sets of X and Y. Thus $\pi(K) < \infty$ for every compact G_δ sets of
X × Y. Proves that $\pi = \mu \times v$ is a Baire measure on X × Y.

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